Microtransition Cascades to Percolation

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(Received 3 October 2013; revised manuscript received 20 March 2014; published 18 April 2014)

We report the discovery of a discrete hierarchy of microtransitions occurring in models of continuous and discontinuous percolation. The precursory microtransitions allow us to target almost deterministically the location of the transition point to global connectivity. This extends to the class of intrinsically stochastic processes the possibility to use warning signals anticipating phase transitions in complex systems.

DOI: 10.1103/PhysRevLett.112.155701 PACS numbers: 64.60.aq, 64.60.ah

Introduction.—Percolation is a pervasive concept [1], which has applications in a wide variety of natural, technological, and social systems [2–7], ranging from conductivity of composite materials [8,9] and polymerizations [10] to epidemic spreading [11–13] and information diffusion [14,15]. Across all percolation systems, once the density of links in the networked system exceeds a critical threshold the system undergoes a sudden usually unanticipated transition to global connectivity.

The prediction of tipping points and warning signals that precede a sudden transition have been a subject of high interest in many disciplines. Generalized models, based on deterministic bifurcation dynamics, have been used to predict phase transitions triggered by small fluctuations [16–20]. Here we report on a fundamental property of percolating systems which, in contrast, are dominated by (nondeterministic) large-scale disorder.

Discrete scale invariance (DSI) arises when the scale invariance of an observable $\mathcal{O}(x) \sim x^\alpha$ obeying $\mathcal{O}(\lambda x)/\mathcal{O}(x) = \lambda^\alpha$, is broken such that the scaling relation does not hold for all λ anymore but only for a countable set $\lambda_1, \lambda_2, \ldots$ with a fixed λ being the fundamental scaling ratio of the system and $\lambda_n = \lambda^n$ [21,22]. Here, we unravel both genuine DSI and a generalized form of DSI in percolation, where in the latter the scaling ratio from the exponential is replaced by a scaling law. Analyzing individual events allows us to link these concepts.

Perhaps most importantly, we show that the emergence of global connectivity is announced by microscopic transitions of the largest component, the order parameter, well in advance of the phase transition. We exemplify this for the generalized Bohman-Frieze-Wormald (BFW) model of genuinely discontinuous percolation [23,24], classic continuous percolation [1], and globally competitive percolation [25]. This suggests the universality of our findings.

Discontinuous percolation.—The generalized BFW model is tailored to investigate discontinuous percolation transitions resulting from suppressing the growth of the largest component [23], as characteristic of *explosive* percolation. The process is initialized with N isolated nodes and a cap set to $k = 2$ specifying the maximally allowed cluster size (a cluster is a set of linked nodes). Links are sampled one at a time, uniformly at random from the complete network. If a link would lead to the formation of a component of size less than or equal to k it is accepted. Otherwise, the link is rejected provided that the fraction of accepted links is greater than or equal to a function $g(k) = \alpha + (2k)^{-1/2}$, where α is a tunable parameter. Once rejecting a link would lead to the fraction of accepted edges dropping below $g(k)$, then $k \to k + 1$ and the link is reexamined. This continues until either k has increased sufficiently that the link can be accepted, or $g(k)$ becomes sufficiently small that the link can be rejected. (see Supplemental Material [26] for more details.) Tuning the control parameter α allows for controlling the type and position of the phase transition, as well as the number of giant components that abruptly emerge [23,32]. Figure 1 shows the typical evolution of the relative size of the largest component C_1/N as a function of the link density p (i.e., number of links per node) for $\alpha = 0.1$, 0.3, 0.6.

The exact size of the largest component for a given link density may depend on the realization. However, in traditional percolation in the thermodynamic limit the order parameter, C_1/N, is believed to be globally continuous and thus not fluctuating—except at the phase transition points [1,34–36]. In contrast, we next demonstrate that the BFW model exhibits peaks in the relative variance $\mathcal{R}_v$, well *before* the phase transition, which importantly do not disappear in the thermodynamic limit, and moreover, announce the phase transition. The relative variance of an order parameter $\mathcal{O}$, such as the total magnetization $\mathcal{O} = \mathcal{M}$, or the relative size of the largest component $\mathcal{O} = C_1/N$, is defined as

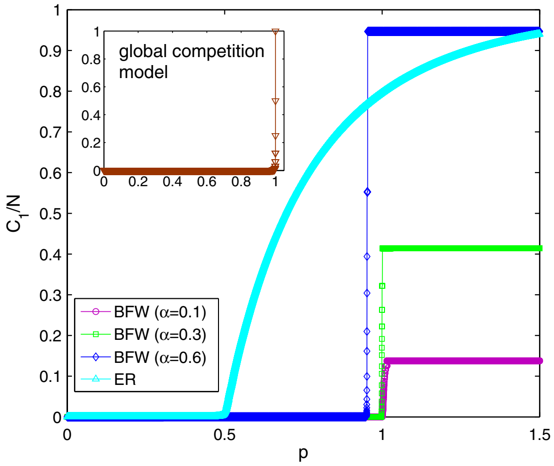

FIG. 1 (color online). Discontinuous BFW percolation. A typical realization of the relative size of the largest component C_1/N as a function of link density p for the BFW model with $\alpha = 0.1$, 0.3, 0.6, and for the continuous ER model. Inset: Discontinuous global competition model. System size $N = 10^6$.

$$\mathcal{R}_v = \frac{\langle \mathcal{O} - \langle \mathcal{O} \rangle \rangle^2}{\langle \mathcal{O} \rangle^2}, \tag{1}$$

where $\langle \cdots \rangle$ denotes ensemble averaging.

Microtransition cascades to percolation.—Figure 2(a) shows sharp peaks in $\mathcal{R}_v$ well in advance of p_c for the BFW model with $\alpha = 0.6$ (figures for $\alpha = 0.1$, 0.3 are in the Supplemental Material [26]). This is unexpected as suggested from comparing Fig. 2(a), with the $\mathcal{R}_v$ plot for the Erdős-Rényi (ER) model [1] shown in the inset in Fig. 2(b). In the BFW model we observe not only the standard transition to global connectivity, which is a micro-macro transition, $C_1 : o(N) \to O(N)$ at $p = p_c$, but as well micro-micro transitions, $C_1 \to C_1 + 1$ causing sharp jumps well before the emergence of global connectivity [Fig. 2(b)]. Importantly, for increasing system size, the peaks become sharper, their positions converge to a well-defined set, and peak heights are independent of system size; see Fig. 2 and the Supplemental Material, Figs. S1–S5 [26].

We calculate the height of the $\mathcal{R}_v$ peaks, for jumps $C_1 \to C_1 + 1$, where the ith jump corresponds to C_1 increasing from $i \to i + 1$ at link density p_i. (The jump $1 \to 2$ occurs always when the first link is added; thus, no peak of R_v is observed then.) We estimate the maximum of R_v for the ith jump by assuming that for a fraction q_i of the realizations $C_1 \to C_1 + 1$, while C_1 for a fraction $1 - q_i$ of the realizations has not increased. Hence, from Eq. (1) we obtain

$$\mathcal{R}_v(p_i) = \frac{q_i(1 - q_i)}{(i + q_i)^2} \quad \text{with} \quad q_i = \frac{i}{2i + 1}, \tag{2}$$

where the q_i's satisfy $(\partial \mathcal{R}_v / \partial q_i) = 0$. From Eq. (2), we find $q_4 = 4/9$ $q_5 = 5/11$, and $q_6 = 6/13$, and that

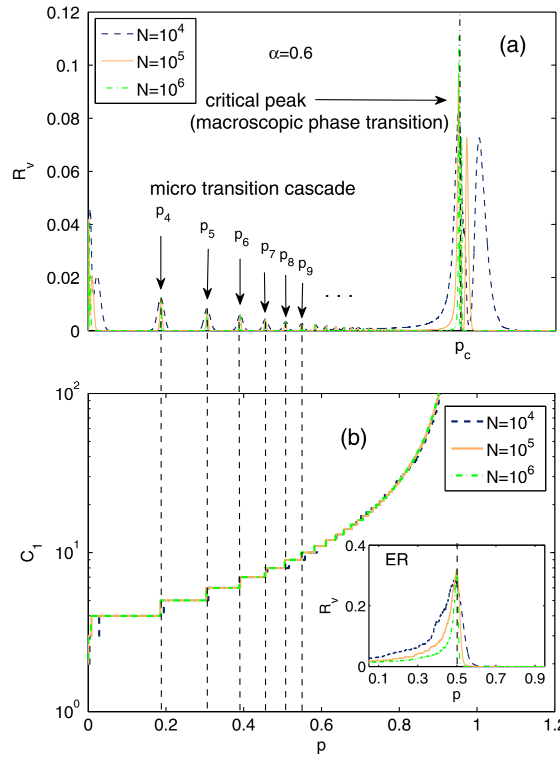

FIG. 2 (color online). Microtransition cascade to percolation in the BFW model. (a) Relative variance R_v versus link density p showing sharp microtransitions before p_c. Peaks after p_c result from unstable giant components, discussed elsewhere [33]. (b) The typical evolution (and collapse) of C_1 versus p, showing jumps when $C_1 \to C_1 + 1$. Inset of (b): $\mathcal{R}_v$ versus p for continuous ER percolation, shown for three different system sizes. This reveals a *spectrum* of microresonances before $p_c = 1/2$, that, in contrast to the BFW model, disappears as $N \to \infty$. All data shown are the average over 1000 realizations.

$\mathcal{R}_v(p_4) \approx 0.0125$, $\mathcal{R}_v(p_5) \approx 0.0083$, $\mathcal{R}_v(p_6) \approx 0.0060$ for the $o(N)$ transitions $4 \to 5$, $5 \to 6$, and $6 \to 7$, respectively. These predictions are well supported by numerics; see the Supplemental Material [26].

Analyzing additional peaks as shown in Fig. 3(a) suggests a scaling law of the relative peak positions

$$\frac{p_{i+1} - p_i}{p_i} \approx \log\left(\frac{p_{i+1}}{p_i}\right) = A i^{-b}, \qquad i \gg 1, \tag{3}$$

with b close to 2, slightly depending on α, for some $A > 0$.

We infer p_∞ from Eq. (3) (see the Supplemental Material [26] for details) and find that $p_\infty = p_c = 0.940$, 0.998, 0.999 for $\alpha = 0.6$, 0.3, 0.1 respectively, which agree exactly with the values of p_c obtained from direct simulation of the BFW model. (See the Supplemental Material [26] for p_∞ values obtained for additional α values.) In fact, the inset of Fig. 3(a) shows that $p_c - p_i < 0.01$ when $i > 600$ for $\alpha = 0.1$, 0.3, 0.6. Thus we find here that the positions of the microtransitions announce the phase transition.

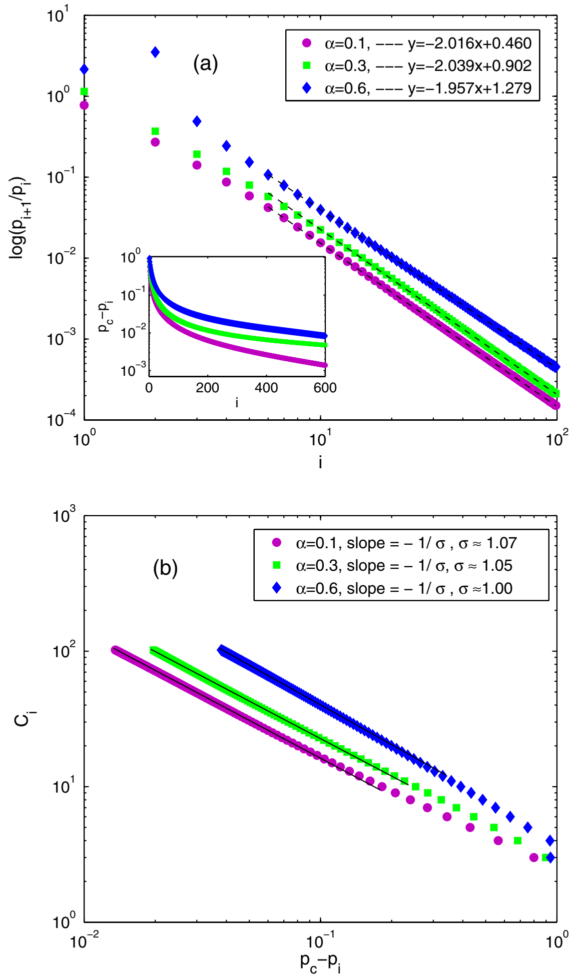

FIG. 3 (color online). Scaling laws and convergence to p_c for the BFW model. (a) The positions p_i of the microtransitions are well fitted by Eq. (3) for $\alpha = 0.1, 0.3, 0.6$. Note that we display a log-log plot suggesting $\log(p_{i+1}/p_i) = Ai^{-b}$. Inset of (a): Evidence for $p_i \to p_c$ for $\alpha = 0.1, 0.3, 0.6$ and $N = 10^7$. (b) Exponent σ defined in Eq. (8) for $\alpha = 0.1, 0.3, 0.6$ and $N = 10^7$.

Discrete scale invariance in percolation.—Next we show that a percolation model with global competition for link addition exhibits a discrete scale invariance that underlies the observed cascade to percolation.

Start with N isolated nodes. At each step connect the two smallest clusters in the system (if there are multiple choices, throw a fair dice to choose among the equivalent cluster pairs) [25,37]. In this model all possible links compete for addition. Thus it is the limiting case $m \to \infty$ of the original explosive percolation models from Ref. [38], where at each step a fixed number of m links compete for addition [25,37]. The global competition suppresses transitions different from *doubling* transitions $C_1 \to 2C_1$ resulting in $p_c = 1$. For $N \gg 1$ fixed, these occur at $p_n = (2^n - 1)/2^n$, n integer [25], and hence

$$p_n = p_c - 2^{-n}, \qquad n \geq 0. \tag{4}$$

As a result, the doubling transitions announce the percolation transition as $p_n \to p_c$ for $n \to \infty$. This is a signature of discrete scale invariance (DSI) [21,39] as we can rewrite Eq. (4) to

$$\frac{p_c - p_{n+1}}{p_c - p_n} = 1/\lambda, \qquad C_1(p_{n+1}) = \lambda C_1(p_n), \tag{5}$$

with the discrete scaling factor $\lambda = 2$.

The DSI can be broken when the system stochastically deviates from the strict size doubling rule, as generically given in percolation and other disordered systems [22]. We thus consider jumps from any size $C_1 \leq i$ to precisely $C_1 = i + 1$. The index transformation $i = 2^{n+1} - 1$ formally breaks the genuine DSI and suggests, using Eq. (4), the transition positions $p_i = 1 - 2/(i + 1)$.

The relative positions of the transitions then read

$$\frac{p_{i+1} - p_i}{p_i} \sim i^{-b}, \qquad \text{for } i \gg 1, \tag{6}$$

with $b = 2$, which agrees well with the scaling law Eq. (3). It is easy to see that any transformation of type $n \to \alpha \log(\beta i + \gamma)$, with constants $\alpha, \beta > 0$ and any γ gives the same qualitative result.

Relation to cutoff critical exponent.—Next we demonstrate that microtransitions also announce the phase transition well in advance for continuous percolation. In continuous percolation as $p \to p_c$, from below ($p < p_c$), the emergence of the giant cluster is characterized by

$$C_1 \sim (p_c - p)^{-\frac{1}{\sigma}}, \tag{7}$$

where σ is the cutoff critical exponent that, given strong disorder, is related to the correlation exponent ν and the fractal dimension d_f via $\sigma = (1/\nu d_f)$ [1,27].

We estimate the positions of the microtransitions at p_i from Eq. (7) for $C_1 = i + 1$ and $p = p_i$. Solving for p_i gives

$$p_i = p_c - A(i + 1)^{-\sigma}, \tag{8}$$

with some prefactor $A > 0$. From Eq. (8) we find

$$\frac{p_{i+1} - p_i}{p_i} \approx \frac{A[(i+1)^{-\sigma} - (i+2)^{-\sigma}]}{p_c} \sim i^{-(1+\sigma)} \quad \text{for } i \gg 1. \tag{9}$$

This equation predicts for *any* phase transition characterized by the exponent σ a cascade defined by Eqs. (6) and (9) with exponent $b = 1 + \sigma$.

Above the percolation upper critical dimension, and thus for ER percolation, the set of critical percolation exponents are known, $\sigma = \nu = 1/2$, $d_f = 4$ [1]. For ER percolation, Eq. (8) is well supported by numerics; see Fig. 4. Further,

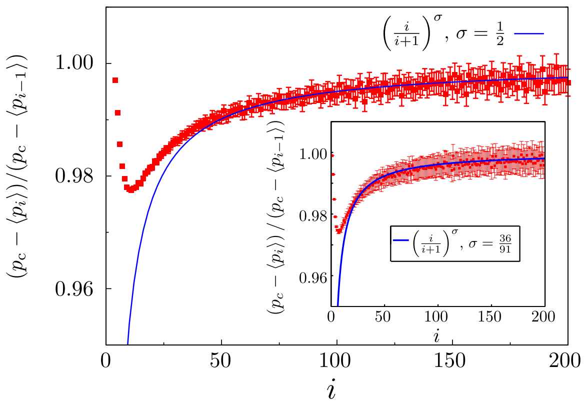

FIG. 4 (color online). Scaling relations for continuous percolation. Numerical evidence for the prediction $[(p_c - p_i)/(p_c - p_{i-1})] \sim [i/(i+1)]^{\sigma}$ from Eq. (8), for the ER model ($\sigma = 1/2$, $N = 2^{25}$, 30 000 realizations) and 2D lattice ($\sigma = 36/91$, $N = 1024 \times 1024$, 30 000 realizations).

numerics for 2D site percolation, where $\sigma = (1/\nu d_f) = 1/(4/3 \times 91/48) \approx 0.396$ is known from theory [1], well supports our prediction (see inset in Fig. 4).

Specifically, we define p_i as the position of the microtransition of the type $C_1 : x \to i + 1 (x \le i)$. Since for a given realization a jump of this type and thus p_i may not exist, to obtain $\langle p_i \rangle$ in Fig. 4, we average for each i over all realizations where $C_1 : x \to i + 1 (x \le i)$ do occur and p_i is well defined,

$$\langle p_i \rangle := \langle \arg_{\exists (i, x \le i)} \{ C_1(p) : x \to i + 1 \} \rangle. \tag{10}$$

In contrast, for fixed N, most pronounced close to the origin at $p = 0$, microtransitions localize but ensemble averaging “blurs out" peaks in $\mathcal{R}_v$ for larger values of p (see Figs. S7 and S8 in the Supplemental Material [26]).

For the BFW model we find the exponent σ, slightly depending on α, close to unity; see Fig. 3(b). This result is in agreement with Eq. (6), predicting $b \approx 2$, and with the numerics shown in Fig. 3(a).

For the globally competitive percolation model we calculate for $p < p_c$ [25,40,41]

$$C_1 = \frac{N}{N - L} = \frac{1}{1 - p} = (p_c - p)^{-\frac{1}{\sigma}}, \quad p = L/N, \tag{11}$$

with $\sigma = 1$, which is an exact result.

Further, from Eq. (7) we calculate the relative positions for transitions of type $C_1 \to nC_1$, for $n > 1$ fixed,

$$\frac{p_c - p_{ni}}{p_c - p_i} \to n^{-\sigma} = n^{1-b} = : 1/\lambda^{(n)}, \quad \text{for } i \to \infty. \tag{12}$$

Equation (12) describes a family of microtransition scaling relations parametrized by n.

We can also turn Eq. (12) around for predicting p_c. For the ER model we find for $n = 2$, $\lambda^{(2)} = 2^{\sigma} = \sqrt{2}$ and

$$p_c = \lim_{i \to \infty} \frac{\lambda^{(2)} p_{2i} - p_i}{\lambda^{(2)} - 1}. \tag{13}$$

Numerical evaluation of Eq. (13) suggests $p_c = 0.4996$ for $i = 128$, which is close to the exact value $p_c = 0.5$ [1].

Conclusion.—We have established the appearance of well-defined peaks in the subcritical regime for standard processes of continuous and discontinuous percolation. The cause of those resonances in the relative fluctuation function are microtransitions of type $o(N) \to o(N)$ that generically announce the percolation phase transition well in advance of p_c. Therefore, genuine peaks in the relative variance do not necessarily indicate a phase transition point, as it is commonly exploited for characterization of the phase transition point in classical and quantum critical systems [27]. We have discovered an overlooked phenomenon, microtransition cascades in percolation, which as shown here can result from a (generalized) discrete scale invariance of the order parameter at and before criticality.

Globally competitive percolation displays genuine discrete scale invariance where the positions of the microtransitions are characterized by powers of the single fundamental scaling factor $\lambda = 2$. This results from a single route of doubling transitions of the order parameter, for large finite systems.

In contrast, we have demonstrated that systems with strong disorder display multiple microtransition cascades to percolation that are not characterized by a single scaling factor but by a set of scaling relations, exemplified for percolation. The simplest subset of these scaling relations describes transitions $C_1 \to nC_1$, $n \ge 2$ integer, which occur at localized positions, for large finite systems. We call this phenomenon generalized discrete scale invariance in percolation.

We have established a novel type of finite size scaling laws which crucially characterize percolation. As our arguments are independent of the percolation process and the system size, for any $N < \infty$ there necessarily exist cascades to percolation imprinted both in the order parameter and its relative variance. Exemplified for a well-studied discontinuous percolation process, we have shown that these cascades can even survive the thermodynamic limit.

Continuous percolation exhibits a continuous power law divergence at p_c that does not show any localized peaks in the relative variance in the thermodynamic limit. In contrast, for fixed N, microtransitions do localize albeit ensemble averaging blurs out peaks in $\mathcal{R}_v$. Ensemble averaging in accord with Eq. (10), however, robustly unravels the discrete hierarchy and thus overcomes the effect of blurring.

We find DSI and its (exponential or power law) scaling laws from a nontrivial exponentiation ($C_1 \to \lambda C_1$ at $p_i \to p_{i+1}$) of a discrete translational invariance resulting from the discreteness of the network, or lattice [21].

Hence, a percolation phase transition can be anticipated by inferring information from ensemble averaged microscopic state changes of the order parameter well in advance of the transition point. Thus we are able to extend the possibility of early warning signals to classes of stochastic dynamics. Future work must establish if these findings will open new avenues for the prediction of phase transitions unrelated to percolation.

This work was funded by the 973 National Basic Research Program of China under Grants No. 2013CB329602 and No. 2012CB316303, the National Natural Science Foundation of China under Grants No. 61232010 and No. 11305219, National High Technology Research and Development Program of China under Grant No. 2014AA015103, the Defense Threat Reduction Agency Basic Research Grant No. HDTRA1-10-1-0088, and the Army Research Laboratory Cooperative Agreement W911NF-09-2-0053 and W911NF-13-1-0340.

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