Chap 2.4 Sequences & summations

- A sequence is an ordered list (elements can repeat)

$$S_1 = \{a_0, a_1, a_2, \dots a_m\}$$

$$S_2 = \{b_0, b_1, b_2, \dots\}$$

let a_j denote the j'th element in the list.

Typically $j \in \mathbb{N} = \{0, 1, 2, 3, \dots\}$

Sometimes $j \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$

Question: How do we find the pattern (i.e. the generating func) that gave rise to the sequence? (i.e. IQ tests)

Example sequences:

$$S_1 = \{1, 1/2, 1/4, 1/8, 1/16, \dots\}$$

$$S_2 = \{1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \dots\}$$

$$S_3 = \{5, 11, 17, 23, 29, \dots\}$$

Find a formula to describe the sequence (may be several formulas that work)

e.g. $S_1 = \{a_j\}$ where $a_j = 1/2^j$ for $j \in \mathbb{N}$

$S_3 = \{a_j\}$ where $a_0 = 5$, $a_j = a_{j-1} + 6$ for $j \in \mathbb{Z}^+$

also $a_j = 5 + j6$

2. An arithmetic progression

$S = \{a_j\}$ where $\boxed{a_j = a + jd}$ for $j \in \mathbb{N}$

(e.g. S_3 "add 6" is such an example).

$a + jd$ is the discrete analogue to a function with linear growth

$f(x) = a + dx$ in calculus $x \in \mathbb{R}$

If $x \in \mathbb{N}$ $f(x) = a + dx$ sampled discretely.

3. A geometric progression

$S = \{a_j\}$ where $\boxed{a_j = ar^j}$ for $j \in \mathbb{N}$, $r \in \mathbb{R}$

ar^j is the discrete analogue to a function w/ exponential growth

$f(x) = ar^x$ in calculus $x \in \mathbb{R}$

If $x \in \mathbb{N}$, $f(x) = ar^x$ is sampled discretely.

e.g. let $a = 2$, $r = 1$; $S = \{2, 2, 2, 2, \ldots\}$

let $a = 1$, $r = 2$; $S = \{1, 2, 4, 8, \ldots\}$

let $a = 1$, $r = 1/2$; $S = \{1, 1/2, 1/4, 1/8, 1/16, \ldots\}$

4. <u>Summations of series</u> (Analogue of integrating a continuous func).

e.g. Sum the first 5 elements

$$a_0 + a_1 + a_2 + a_3 + a_4 =: \sum_{i=0}^{4} a_i$$

More abstract:

$$a_m + a_{m+1} + a_{m+2} + \cdots + a_n =: \sum_{i=m}^{n} a_i = \sum_{j=m}^{n} a_j$$

index name is flexible.

Index is flexible:

let $l = j+1$

$$\text{then} \quad \sum_{j=m}^{n} a_j = \sum_{l=m+1}^{n+1} a_l$$

recall $l = j+1$

$j = l-1$

* We will take advantage of index fluidity.

Useful sum; sum a sequence of integers.

$$S_n =: \sum_{j=1}^{n} j = 1 + 2 + 3 + \cdots + n$$

How to evaluate?

Concrete example:

$$S_6 = \sum_{j=1}^{6} j = 1 + 2 + 3 + 4 + 5 + 6 = 7\left(\frac{6}{2}\right)$$

Generalize

$$S_n = \sum_{j=1}^{n} j = 1 + 2 + \cdots + n-1 + n = \underbrace{(n+1)}_{\text{value of the sum of the pair}} \underbrace{\left(\frac{n}{2}\right)}_{\text{\# of pairs}}$$

$$\boxed{S_n = \sum_{j=1}^{n} j = \frac{(n+1)\, n}{2}}$$

Show this holds for n even and n odd.

If n is even:

$$1 + 2 + 3 + \cdots + (n-1) + n = \underbrace{(n+1)}_{\text{sum of pairs}} \underbrace{\left(\frac{n}{2}\right)}_{\text{\# of pairs}}$$

If n is odd:

e.g. $S_7 = 1 + 2 + 3 + 4 + 5 + 6 + 7$

odd pairing it has the value of $(n+1)/2$

If n is odd, generally.

$$S_n = 1 + 2 + 3 + \cdots \frac{(n+1)}{2} + \cdots + (n-1) + n$$

· sum of "big/small" integer pairs is $(n+1)$

· We have $(n-1)/2$ pairs.

$$S_n = \underbrace{(n+1)}_{\text{sum of pairs}} \underbrace{(n-1)/2}_{\text{\# of pairs}} + \underbrace{\frac{(n+1)}{2}}_{\text{middle-term}}$$

$$= \left(\frac{n+1}{2}\right)\left[(n-1)+1\right] = \frac{n(n+1)}{2}$$

we have proven that

$$\boxed{S_n = \sum_{j=1}^{n} j = \frac{n(n+1)}{2}}$$

Likewise we can show

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

prove by induction later on.

6. Summing a geometric series

(e.g. integrating an exponential func)

$$S_n = \sum_{j=0}^{n} a r^j$$

• If $r = 1$, $S_n = \sum_{j=0}^{n} a$ (since $1^k = 1$)

$$S_n = \underbrace{a + a + \cdots + a}_{(n+1) \text{ times}} = (n+1)a.$$

• If $r \neq 1$ more complicated.
Direct sum is difficult.

Use a "clever" trick and sum rS_n instead:

$$rS_n = r\sum_{j=0}^{n} ar^j = \sum_{j=0}^{n} ar^{j+1}$$

Let $k = j+1$.
(if $k=1$, $j=0$)

$$rS_n = \sum_{k=1}^{n+1} ar^k$$

$$= \sum_{k=1}^{n} ar^k + ar^{n+1}$$

(used the fact that $\sum_{j=1}^{n} a_j = \sum_{j=1}^{n-1} a_j + a_n$)

Use flexible indices again:
(Use the fact $\sum_{j=1}^{n} a_j = \sum_{j=0}^{n} a_j - a_0$)

$$r S_n = \sum_{k=1}^{n} ar^k + ar^{n+1}$$

$$= \sum_{k=0}^{n} ar^k - ar^0 + ar^{n+1}$$

$$= \underbrace{\sum_{k=0}^{n} ar^k}_{= S_n \text{ the geometric series.}} + ar^{n+1} - a$$

$$r S_n = S_n + (ar^{n+1} - a)$$

$$S_n (r-1) = (ar^{n+1} - a)$$

$$\boxed{S_n = \frac{ar^{n+1} - a}{(r-1)}} \quad \text{if } r \neq 1.$$

Geometric series

$$S_n = \begin{cases} (n+1)a & \text{if } r = 1 \\ (ar^{n+1} - a)/(r-1) & \text{if } r \neq 1. \end{cases}$$

7. Sum an arithmetic progression.

$$S_n = \sum_{j=0}^{n} (a + jd)$$

$$= \underbrace{\sum_{j=0}^{n} a}_{\substack{\text{geometric} \\ \text{series w/ } r=1}} + \underbrace{\sum_{j=0}^{n} jd}$$

Equal to $d \sum_{j=0}^{n} j = d \frac{n(n+1)}{2}$

sum $= (n+1)a$.

(note $\sum_{j=1}^{n} j = \sum_{j=0}^{n} j$)

$$S_n = (n+1)a + \frac{d\,n(n+1)}{2}$$

8. Useful sums:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

why $e^x \simeq 1 + x$ for $|x| << 1$ and $x \in \mathbb{R}$