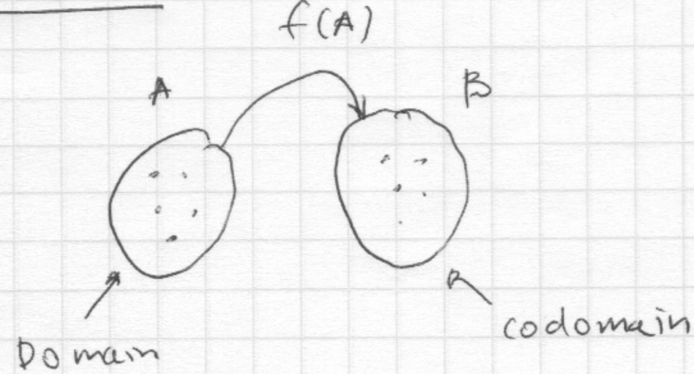
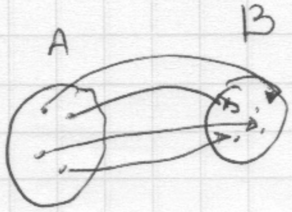
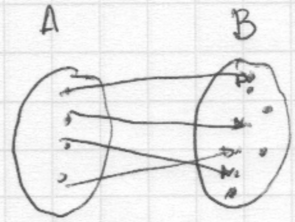
Lecture 6

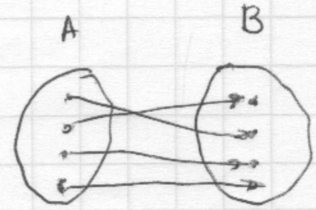
Review functions

$f: A \to B$









function

Every $a \in A$ must have one out-arrow

One-to-one (injective)

Each $b \in B$ has only one in arrow

$|\text{codomain}| \geq |\text{Domain}|$

Onto (surjective)

Each $b \in B$ has at least one in arrow

$|\text{codomain}| = |\text{Range}|$

One-to-one plus onto (bijection) then f^{-1} exists.

$|\text{codomain}| = |\text{range}| = |\text{domain}|$

TODAY: - Rules of inference (slides)

- Methods of proof (written notes).

Remember important info:

- $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$
- $\mathbb{Z} = \{\ldots -2, -1, 0, 1, 2, \ldots\}$
- $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$
- $\mathbb{R}$ the real numbers

$\mathbb{N} \subset \mathbb{Z}$, $\mathbb{Z}^+ \subset \mathbb{N}$, $\mathbb{N}, \mathbb{Z}, \mathbb{Z}^+ \subset \mathbb{R}$

- $p \to q$ "if p then q"
  "p implies q" / "p is sufficient for q"
  "q is necessary for p"
  "p <u>only</u> if q" ← tricky English

- $p \leftrightarrow q$ "p is necessary and sufficient for q"
  "q " " " " " p"
  "p if and only if q"
  "q " " " " p"

New facts we will use in our proofs

- An integer n is <u>even</u> if there exist an integer k such that $n = 2k$.

- An integer n is <u>odd</u> if there exists an integer k s.t. $n = 2k+1$

- Any integer is either even or odd (not both)

- A real number r is <u>rational</u> if there are integers $p \neq q$ s.t. $r = P/q$, where p and q have no common factors

  Otherwise r is <u>irrational</u> (e.g. π, $\sqrt{2}$, e)

- A <u>prime #</u> has no factors (other than itself and 1)

  If not prime, it is <u>composite</u>.

Methods of proof

1) <u>Direct proof</u> of $p \to q$ or $P(n) \to Q(n)$

Assume p is true, use axioms, defn's, lemmas, theorems, to show q must also be true.

Example Direct Proof!

If $\underbrace{n \text{ is an odd integer}}_{P(n)}$, then $\underbrace{n^2 \text{ is odd}}_{Q(n)}$.

Prove $\forall n \in \mathbb{Z}\,(P(n) \to Q(n))$

Proof: $n = 2k+1$ (assume $P(n) = T$, defn)

$$\left.\begin{aligned} n^2 = (2k+1)(2k+1) &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \\ &= 2j + 1 \quad \text{where } \cancel{j = (k^2 + k)} \\ & \qquad j = (2k^2 + 2k) \end{aligned}\right\} \text{arith.}$$

$\therefore$ n^2 is odd since $n^2 = 2j + \underline{1}$.

2) <u>Indirect proof</u>; usually <u>contra positive</u>.

Prove $p \to q$ is true by showing $\neg q \to \neg p$ is true.

Assume $\neg q$, together w/ axioms, defn's, lemmas, etc to show $\neg p$ must also be true.

Prove if $\underbrace{3n+2 \text{ is odd}}_{P(n)}$ then $\underbrace{n \text{ is odd}}_{Q(n)}$

$$P(n) \longrightarrow Q(n)$$

Prove $\forall n \in \mathbb{Z} \; (P(n) \rightarrow Q(n))$

Try direct proof:

$3n+2 = 2j+1$ (def'n of odd)

$n = \dfrac{2j-1}{3}$ ... what do I do now??

Try contrapositive: $\neg Q(n) \rightarrow \neg P(n)$

$n = 2k$ (assume $\neg Q(n)$)

$3n+2 = 3(2k)+2 = 6k+2 = 2(3k+1) = 2j$ (arthmitec)

$\therefore$ $3n+2 = 2j$ is even. (where $j = 3k+1$)

Showed $\neg Q(n) \rightarrow \neg P(n)$ for all $n \in \mathbb{Z}$

- Prove if n is even then n^2 is even.

$$\forall n \in \mathbb{Z}\, (P(n) \to Q(n))$$

Direct:

$$n = 2k$$

$$n^2 = 4k^2 = 2(2k^2) = 2j$$

$\therefore n^2 = 2j$ is even.

- Prove if n^2 is even then n is even.

Direct proof $n^2 = 2k$ so $n = \sqrt{2k}$ ...???

Contrapositive proof:

$$n = 2k+1$$

$$n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

$\therefore n^2$ is odd

Showed $\neg Q(n) \to \neg P(n)$

3) Proof by contradiction

a) of a proposition p

- Show that $\neg p \rightarrow (r \wedge \neg r)$ for some independent proposition r.
(thus $p = T$)

b) of an implication $p \rightarrow q$

$$\begin{array}{l} p \rightarrow q \\ p \\ \neg q \\ \hline \therefore (q \wedge \neg q) \end{array} \qquad \text{or} \qquad \begin{array}{l} \neg q \rightarrow \neg p \\ \neg q \\ p \\ \hline \therefore (p \wedge \neg p) \end{array}$$

* $\boxed{\text{In other words show that } p \wedge \neg q \rightarrow (p \wedge \neg p) \vee (q \wedge \neg q)}$

- Example contradiction proof of a proposition

Prove $p :=$ "$\sqrt{2}$ is irrational"

$r :=$ "a rational number $l = a/b$ where a, b have no common factor."

Show $\neg p \rightarrow (r \wedge \neg r)$

Proof:

$\sqrt{2}$ is rational (assert $\neg p$)

$\sqrt{2} = a/b$ where a, b satisfy $r = T$

$2 = a^2/b^2$

$a^2 = 2b^2$

$a^2 =$ even # since $a^2 = 2j$ where $j = b^2$

$a =$ even (lemma)

$a = \cancel{\ } 2m$ (defin of even)

$2b^2 = a^2 = 4m^2$

$b^2 = 2m^2$

b^2 is even since $b^2 = 2k$ where $k = m^2$

b is even (lemma)

$\therefore \neg r$ is true since a, b even (they share common factor of 2).

$\neg p \rightarrow (r \wedge \neg r)$