Lecture 4.

- Propositions, $p, q, m, \ldots$ T or F
- Predicates take n-tuples as the subject

$$P(x),\ P(x,y),\ Q(x,y),\ \text{etc.}$$

where x, y, z, etc are elements in a domain of discourse.

- Existential quantifiers, bind the predicates and make them into propositions

e.g. $\exists x$, $\forall x$, $\exists x > 0$, $\neg \forall x$, etc...

- Implication $p \rightarrow q$

consequent/conclusion

Antecedent/hypothesis

$p \rightarrow q$ is only false when $P = T$ and $Q = F$.

(why contrapositive $\neg q \rightarrow \neg p$ is logically equiv).

Typically interested in implications with compound hypothesis and conclusions

e.g. $(s \vee t) \wedge (r \vee u) \rightarrow q$

$$P(x,y) \wedge Q(x,y) \rightarrow Z(x,y)$$

* How to specify which elements x in U satisfy $P(x)$?

Answer TODAY's lecture: Sets & set builder notation

But first let's prove a logical equiv.

Recall De Morgan's laws

1) $\neg(p \wedge q) \equiv \neg p \vee \neg q$

2) $\neg(p \vee q) \equiv \neg p \wedge \neg q$

we can generalize to many variables

1) $\neg(p \wedge q \wedge r \wedge \cdots \wedge z) \equiv \neg p \vee \neg q \vee \neg r \vee \cdots \vee \neg z$

2) $\neg(p \vee q \vee r \vee \cdots \vee z) \equiv \neg p \wedge \neg q \wedge \neg r \wedge \cdots \wedge \neg z$

Let's use this to prove two relations from Lec 3.

slide 28
Asserted:

- $\forall x P(x) \equiv \neg \exists x \neg P(x)$ ← Prove this now
- $\exists x P(x) \equiv \neg \forall x \neg P(x)$ ← on HW2.

Prove: $\forall x P(x) \equiv \neg \exists x \neg P(x)$

Recall $\exists x P(x) \equiv P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n)$

$\forall x P(x) \equiv P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n)$

$\neg \exists x \neg P(x) \equiv \neg(\exists x \neg P(x))$ precedence of not operator

$\equiv \neg(\neg P(x_1) \vee \neg P(x_2) \vee \cdots \vee \neg P(x_n))$ defn $\exists$

$\equiv \neg\neg P(x_1) \wedge \neg\neg P(x_2) \wedge \cdots \wedge \neg\neg P(x_n)$

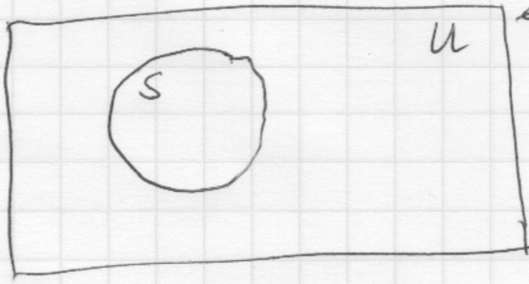
Demorgan's 2

$\equiv P(x) \wedge P(x) \wedge \cdots \wedge P(x_n)$

$\equiv \forall x P(x)$

defn of $\forall x$

Specifying Sets



Domain / universe of discorse.
(e.g. people, integers, real #'s, etc)

Let us denote elements of U

$x_1, x_2, x_3, \ldots x_n$

S is a subset of the x_i's that have some property in common.

Typically S as the set of x_i's for which $P(x) = T$.

Two ways to specify sets

- explicit enumeration

$S = \{x_1, x_2\}$ or $S = \{x_5, x_7, x_9, x_{11}, \ldots\}$

curly braces.

- set builder notation

$S = \{x \mid P(x)\}$ * the collection of elements in U for which $P(x)$ is True.

If x_i is an element is S

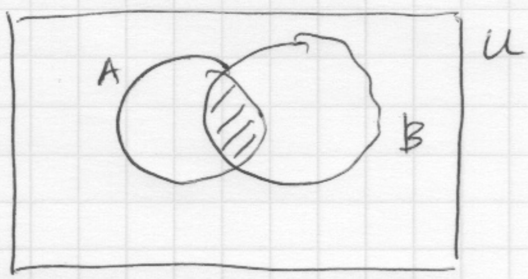
we say $x_i \in S$, Note $x_i \in U$

* we want the collection of x_i's for which $P(x)$ is true, so only each x_i once.

Consider two sets, e.g. $A = \{x \mid P(x)\}$, $B = \{x \mid Q(x)\}$

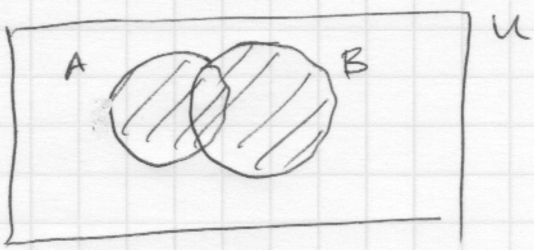
Intersection, union & difference.

- Intersection $A \cap B$ (elements they share in common)



(shaded area)

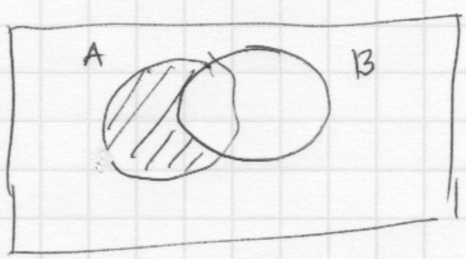
- Union $A \cup B$ (all elements, counted only once)



$$A \cup B = A + B - \underbrace{A \cap B}$$

each element only counts once.
(Inclusion / exclusion principle)

Difference: $A - B$



But set can be complex;

- Sets of sets; e.g. $T = \{\{x_1, x_2\}, \{x_5, x_7, x_9\}, \ldots\}$
- can have multiple variables from different U's

e.g. $T = \{(x,y) \mid P(x) = T \wedge Q(y) = T\}$

Very important sets

- The empty set $\emptyset = \{\}$
- The Natural numbers; $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$
- The Integers, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- The positive integers $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$
- The real numbers $\mathbb{R}$

... onwards to Lec 4 slides ....