

Lec 16 counting = enumerate all configurations.

- Product rule - outcomes multiply

- Sum rule - outcomes add

1. Product rule

Last time # of func's from $|A|=m$ to $|B|=n$

$$\frac{\quad}{n} \frac{\quad}{n-1} \frac{\quad}{n-2} \cdots \frac{\quad}{(n-m+1)}$$

$$\# \text{ choices: } n(n-1)(n-2)\cdots(n-m+1) = \frac{n!}{(n-m)!}$$

$$(n-m)! = (n-m)(n-m-1)\cdots(2)(1)$$

* Note typo at end of Lec 15.

(said $\frac{n!}{m!}$ should say $\frac{n!}{(n-m)!}$)

Sum: Outcome is from a set A_1 or a set A_2
possibilities add.

• Disjoint sets $A_1 \cap A_2 = \{\}$

$$|A_1 \cup A_2| = |A_1| + |A_2|$$

• Not disjoint: Inclusion-exclusion principle:

$$\text{IF } A_1 \cap A_2 \neq \{\} \text{ then } |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

Example:

of bit strings of length 8 that start with
1 or end with 00.

Bit $\in \{0, 1\}$



- # of bit strings starting w/ 1; 2^7 choices
 - # of bit strings ending w/ 00; 2^6 choices
 - # of bit strings starting w/ 1 and ending w/ 00; 2^5 choices
-

$$2^7 + 2^6 - 2^5 = 160$$

e.g. $|A_1| + |A_2| - |A_1 \cap A_2|$

Intuitive example of prod rule & sum rule

Company makes shirts

Scenario 1: 2 M's, 3 W's

$$\# \text{ of styles: } \# \text{ M's} + \# \text{ W's} = 5.$$

Scenario 2: 2 M's, 3 W's, 2 colors for each

$$\# \text{ of styles} \quad \underbrace{2 \cdot 2}_{\# \text{ M's}} + \underbrace{2 \cdot 3}_{\# \text{ W's}} = 10.$$

Scenario 3: 2 M's, 3 W's, 2 colors, 2 sizes.

styles:

$$\underbrace{(2 \cdot 2 \cdot 2)}_{\text{prod rule}} + \underbrace{(3 \cdot 2 \cdot 2)}_{\text{prod rule}} = 20$$

sum rule.

Final technique for counting: Pigeonhole Principle

New topic Handout 12: Permutations & combinations

- A permutation: an ordered list of elements.
(a sequence w/ no repetition)

e.g. $1, 2, 3 \neq 2, 3, 1 \neq 3, 2, 1$ (order matters)

$P(n, r)$ ← the # of distinct permutations given n elements and selecting r of them.

$$\begin{array}{ccccccc} \overline{} & \overline{} & \overline{} & \dots & \overline{} \\ \uparrow & \uparrow & & & \uparrow \\ n & (n-1) & & & (n-r+1) \end{array}$$

$$P(n, r) = \frac{n!}{(n-r)!}$$

$$P(n, n) = \frac{n!}{0!} = n! \quad (n! \text{ permutations of } n \text{ objects})$$

Examples:

a) 10 people, 4 slots $= P(10, 4) = \frac{10!}{6!}$

$$\begin{array}{cccc} \overline{} & \overline{} & \overline{} & \overline{} \\ \uparrow & \uparrow & \uparrow & \uparrow \\ 10 & 9 & 8 & 7 \end{array}$$

b) # of permutations of ABCDEFGH that contain ABC?
1 choice (treat as unit)

6 choices: (ABC), D, E, F, G, H.

$$P(6, 6) = \frac{6!}{0!} = 6!$$

Combination: Order of occurrence does not matter
 (a set not a sequence)
 ↗ combination ↘ permutation

$C(n, r) \equiv \#$ of possible subsets of size r
 that can be selected from a set of n objects.

Consider r objects: — — — — —
 $r!$ possible permutations.

Recall $P(n, r)$

— — — — —
 r slots; $r!$ possible orderings of the set
 of r elements.

$$C(n, r) = \frac{P(n, r)}{r!}$$

$$C(n, r) = \frac{n!}{r! (n-r)!} \quad \text{"n choose r"}$$

$$C(n, r) \equiv \binom{n}{r} \quad \text{"n choose r"}$$

$$C(n, n) = \frac{n!}{n! (0!)} = 1. \quad (\text{Recall } P(n, n) = n!)$$

Prod rule, sumrule, inclusion - exclusion, $P(n,r)$, $C(n,r)$

We can now count fully & do probability.

Domain of application \rightarrow Poker (5 card stud)

Deck of cards:

- 13 possible card types; $\{2, 3, 4, \dots, 10, J, Q, K, A\}$
- 4 possible card suits; $\{\heartsuit, \clubsuit, \diamondsuit, \spadesuit\}$
- # of cards in a deck = $13 \cdot 4 = 52$.

Rules: each player is dealt 5 cards at random and order does not matter.

of distinct 5 card "hands"; $C(52, 5)$

$$C(52, 5) = \frac{52!}{5! 47!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2} = 2,598,960.$$

$$\text{Prob of any particular "hand"} = \frac{1}{2,598,960}.$$

We care about more general classes of hands

e.g.

1 pair

2 pairs

3 of a kind

Full house, etc.

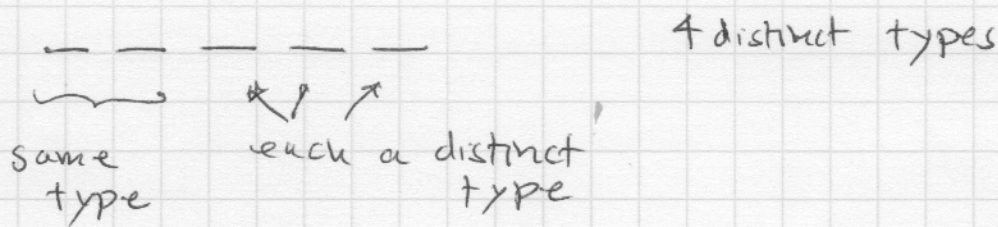
$C(n,r)$ will help us determine the number of all these different classes.

* Depend on the details of the specific class.

(understand the structure of the hand)

Example 1: A pair of cards match, all else distinct.

("having a pair")



General plan:

- (i) # of types needed for the hand
- (ii) # of ways to assign the types
- (iii) # of ways to assign suits

of hands w/ one pair & only one pair

4 distinct types $\rightarrow C(13, 4)$
 1 of those types gets to have a pair $\rightarrow C(4, 1)$
 4 distinct suits \rightarrow pair: $C(4, 2)$
 \rightarrow singles: $C(4, 1)$ each single.

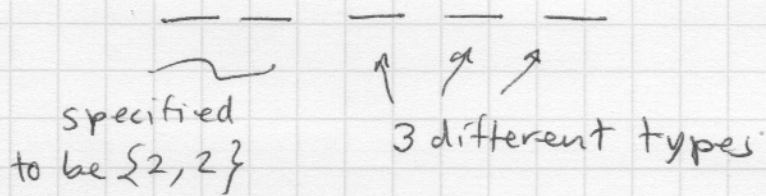
Prod rule.

of pairs: $C(13, 4) C(4, 1) C(4, 2) C(4, 1) C(4, 1) C(4, 1)$

$\underbrace{\hspace{10em}}$ # of types
 $\underbrace{\hspace{10em}}$ which type gets the pair
 $\underbrace{\hspace{10em}}$ suits in the pair
 $\underbrace{\hspace{10em}}$ suits of each single.

Instead of all hands w/ a pair,

how many hands have a pair of 2's?



of types ~~$C(13, 3)$~~ $C(12, 3)$ ← one type went to specify 2 so only 12 types remain.
of suits for pair $C(4, 2)$
of suits for singles $C(4, 1)$

hands w/ a pair of twos:

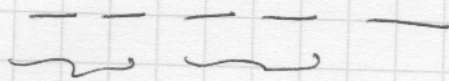
$$C(12, 3) \cdot C(4, 2) \cdot C(4, 1) \cdot C(4, 1) \cdot C(4, 1)$$

types for remaining 3 cards # of suits for the pair of 2's choose suits for last 3 types.

$$\text{Probability of a pair} = \frac{C(13, 4) \cdot C(4, 1) \cdot C(4, 2) \cdot (C(4, 1))^3}{C(52, 5)}$$

$$\text{Prob a pair of 2's} = \frac{C(12, 3) \cdot C(4, 2) \cdot (C(4, 1))^3}{C(52, 5)}$$

Example: Two pairs



3 types in total

$C(13, 3)$ # of types

$C(3, 2)$ # of ways the types are assigned pairs

$C(4, 2)$ # of suits of each pair

$C(4, 1)$ # of suits for remaining card.

of hands, two pairs:

$$C(13, 3) C(3, 2) C(4, 2) C(4, 2) C(4, 1)$$

Another class of problems: choosing committees

• How many ways can we choose a committee of 5 people from 11; $C(11, 5)$

• Consider of the 11 people; 5 women, 6 men

comms w/ no women: $C(6, 5)$

comms w/ exactly 1 woman: $C(5, 1) \cdot C(6, 4)$

" " " 2 " : $C(5, 2) \cdot C(6, 3)$

" " " 3 " : $C(5, 3) \cdot C(6, 2)$

" " " 5 : $C(5, 5)$

Add all these together = $C(11, 5)$