Lec 16 counting = enumerate all configurations.

- Product rule — outcomes multiply
- Sum rule — outcomes add

1. Product rule

Last time # of func's from $|A| = m$ to $|B| = n$

$$\underset{n}{\_\_} \; \underset{n-1}{\_\_} \; \underset{n-2}{\_\_} \; \_\_ \; \_\_ \cdots \underset{(n-m+1)}{\_\_}$$

choices: $n(n-1)(n-2)\cdots(n-m+1) = \dfrac{n!}{(n-m)!}$

$$(n-m)! = (n-m)(n-m-1)\cdots(2)(1)$$

* Note typo at end of Lec 15.
(said $\dfrac{n!}{m!}$ should say $\dfrac{n!}{(n-m)!}$)

Sum: Outcome is from a set A_1 or a set A_2
possibilities add.

- Disjoint sets $A_1 \cap A_2 = \{\}$

$$|A_1 \cup A_2| = |A_1| + |A_2|$$

- Not disjoint: Inclusion exclusion principle:

If $A_1 \cap A_2 \neq \{\}$ then $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

Example:

# of bit strings of length 8 that start with 1 or end with 00.

Bit $\in \{0, 1\}$

_ _ _ _ _ _ _ _

- # of bit strings starting w/ 1; 2^7 choices
- # of bit strings ending w/ 00; 2^6 choices
- # of bit strings starting w/ 1 and ending w/ 00; 2^5 choices

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$$2^7 + 2^6 - 2^5 = 160$$

e.g. $|A_1| + |A_2| - |A_1 \cap A_2|$

Intuitive example of <u>prod rule</u> & <u>sum rule</u>

Company makes shirts

Scenario: 1: 2 M's, 3 W's

# of styles: # M's + # W's = 5.

Scenario 2: 2 M's, 3 W's, 2 colors for each

$$\#\text{ of styles}\quad \underbrace{2\cdot 2}_{\#\text{ M's}} + \underbrace{2\cdot 3}_{\#\text{ W's}} = 10.$$

Scenario 3: 2 M's, 3 W's, 2 colors, 2 sizes.

$$\#\text{ styles:}\quad \underbrace{2\cdot 2\cdot 2}_{\text{prod rule}} + \underbrace{3\cdot 2\cdot 2}_{\text{prod rule}} = 20$$

sum rule.

Final technique for counting: <u>Pigeonhole Principle</u>

New topic Handout 12: <u>Permutations & combinations</u>

- A <u>permutation</u>: an ordered list of elements.
  (a sequence w/ no repitition)

e.g. $1,2,3 \neq 2,3,1 \neq 3,2,1$ (order matters)

$P(n,r) \leftarrow$ the # of distinct permutations given n elements and selecting r of them.

$$\underset{n}{\underset{\uparrow}{\_}}\ \underset{(n-1)}{\underset{\uparrow}{\_}}\ \_\ \_\ \cdots\ \underset{(n-r+1)}{\underset{\uparrow}{\_}}$$

$$\boxed{P(n,r) = \frac{n!}{(n-r)!}}$$

$$P(n,n) = \frac{n!}{0!} = n!$$ ($n!$ permutations of n objects)

Examples:

a) 10 people, 4 slots

$$\underset{10}{\underset{\uparrow}{\_}}\ \underset{9}{\underset{\uparrow}{\_}}\ \underset{8}{\underset{\uparrow}{\_}}\ \underset{7}{\underset{\uparrow}{\_}} = P(10,4) = \frac{10!}{6!}$$

b) # of permutations of $\underbrace{ABC}_{\text{1 choice (treat as unit)}}DEFGH$ that contain ABC?

6 choices: (ABC), D, E, F, G, H.

$$P(6,6) = \frac{6!}{0!} = 6!$$

Combination: Order of occurrence does not matter

(a set not a sequence)

combination   permutation

$C(n,r) \equiv$ # of possible subsets of size r that can be selected from a set of n objects.

Consider r objects: — — — — — — ... —

$r!$ possible permutations.

Recall $P(n,r)$

$$\underbrace{\text{— — — — — ... —}}_{r \text{ slots}}$$

rslots; $r!$ possible orderings of the set of r elements.

$$C(n,r) = \frac{P(n,r)}{r!}$$

$$\boxed{C(n,r) = \frac{n!}{r!\,(n-r)!}}$$ "n choose r"

$$C(n,r) \equiv \binom{n}{r}$$ "n choose r"

$$C(n,n) = \frac{n!}{n!\,(0!)} = 1.$$ (Recall $P(n,n) = n!$)

Prod rule, sum rule, inclusion - exclusion, $P(n,r)$, $C(n,r)$

We can now count fully & do probability.

Domain of application $\longrightarrow$ Poker (5 card stud)

Deck of cards:

- 13 possible card types ; $\{2, 3, 4, \ldots 10, J, Q, K, A\}$
- 4 possible card suits ; {hearts, clubs, diamonds, spades}
- # of cards in a deck $= 13 \cdot 4 = 52$.

Rules: each player is dealt 5 cards at random and order does not matter.

# of distinct 5 card "hands" ; $C(52, 5)$

$$C(52,5) = \frac{52!}{5!\,47!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2}$$

$$= 2,598,960.$$

Prob of any particular "hand" $= \dfrac{1}{2,598,960}$.

We care about more general classes of hands

e.g. 1 pair
2 pairs
3 of a kind
Full house, etc.

C(n,r) will help us determine the number of all these different classes.

* Depend on the details of the specific class.

(understand the structure of the hand)

Example 1: A pair of cards match, all else distinct.

("having a pair")

__ __ __ __ __ 4 distinct types

same type

each a distinct type

> General plan: (i) # of types needed for the hand
>
> (ii) # of ways to assign the types
>
> (iii) # of ways to assign suits

# of hands w/ one pair & only one pair

4 distinct types → $C(13,4)$

1 of those types gets to have a pair → $C(4,1)$

4 distinct suits

→ pair: $C(4,2)$

→ singles: $C(4,1)$ each single.

Prod rule.

# of pairs: $\underbrace{C(13,4)}_{\text{\# of types}}\ \underbrace{C(4,1)}_{\text{which type gets the pair}}\ \underbrace{C(4,2)}_{\text{suits in the pair}}\ \underbrace{C(4,1)\,C(4,1)\,C(4,1)}_{\text{suits of each single.}}$

Instead of all hands w/ a pair,

how many hands have a pair of 2's?

$$\underbrace{\_\ \_}_{\text{specified to be } \{2,2\}}\quad \underbrace{\_\ \_\ \_}_{\text{3 different types}}$$

\# of types ~~C(13,3)~~ C(12,3) ← one type went to specify 2 so only 12 types remain.

\# of suits for pair C(4,2)

\# of suits for singles C(4,1)

\# hands w/ a pair of twos:

$$\underbrace{C(12,3)}_{\substack{\text{types}\\ \text{for remaining}\\ \text{3 cards}}} \cdot \underbrace{C(4,2)}_{\substack{\text{\# of}\\ \text{suits}\\ \text{for}\\ \text{the pair of 2's}}} \cdot \underbrace{C(4,1) \cdot C(4,1) \cdot C(4,1)}_{\substack{\text{choose suits for}\\ \text{last 3 types.}}}$$

Probability of a <u>pair</u> $= \dfrac{C(13,4) \cdot C(4,1) \cdot C(4,2)\left(\cdot\, C(4,1)\right)^3}{C(52,5)}$

Prob a pair of 2's: $= \dfrac{C(12,3) \cdot C(4,2) \cdot \left(C(4,1)\right)^3}{C(52,5)}$

Example: Two pairs

$\underbrace{\_\ \_}\ \underbrace{\_\ \_}\ \_$ 3 types in total

$C(13,3)$ # of types

$C(3,2)$ # of ways the types are assign pairs

$C(4,2)$ # of suits of each pair

$C(4,1)$ # of suits for remaining card.

# of hands, two pairs: $C(13,3)\,C(3,2)\,C(4,2)\,C(4,2)\,C(4,1)$

Another class of problems: choosing committees

- How many ways can we choose a committe of 5 people form 11 ; $C(11,5)$

- consider of the 11 people; 5 women, 6 men

cmmts w/ no women : $C(6,5)$

cmmt's w/ exacty 1 women: $C(5,1) \cdot C(6,4)$

" " " 2 " : $C(5,2) \cdot C(6,3)$

" " 3 " : $C(5,3) \cdot C(6,2)$

⋮

" " 5 : $C(5,5)$

Add all these together $= C(11,5)$