Lec 14. Proof by Induction

Proof technique for proving propositions $P(n)$
dbount $n \in \mathbb{N}, \mathbb{Z}^+$

Three steps:

1) Assume $P(n)$ (all defns & associated axioms)

2) Use the defn's of $P(n)$ to show
that if $P(n)$ is true, then $P(n+1)$ is true.
Direct proof $P(n) \to P(n+1)$.

3) Show $P(n)$ holds for some smallest
element $\mathbb{Z}^+$ or $\mathbb{N}$, call it n_{min}.

$\therefore \ \forall n \geq n_{min} \quad P(n) = T.$

Typically usefol for:

- Summations
- Inequalities
- Division
- Sets
- Algoritms.

Proof by induction for summations

(General technique).

Example summations

$$S_n = \sum_{i=1}^{n} i \quad , \quad S_{geo}(n) = \sum_{i=0}^{n} ar^i \quad , \quad etc. . .$$

Hypothesis $P(n) := S_n = f(n)$

raw sum

simple, closed-form, eqn.

Need to show $P(n) \rightarrow P(n+1)$

i.e. if $S_n = f(n)$ then $S_{n+1} = f(n+1)$

Steps:

- Write S_{n+1} and execute algebra to make it expressed in terms of S_n.
- Substitute $f(n)$ for S_n (i.e. assume $P(n)$)
- Use algebra to make r.h.s. in terms $f(n+1)$.

For example. Prove $P(n) \rightarrow P(n+1)$

where $P(n) := \sum_{i=1}^{n} (2i-1) = n^2$

- write S_{n+1} & show it in terms of S_n

$$\sum_{i=1}^{n+1} (2i-1) = \underbrace{\sum_{i=1}^{n} (2i-1)}_{P(n)} + \left(2(n+1)-1\right)$$

- Substitute $f(n)$ for $P(n)$

$$\sum_{i=1}^{n+1} (2i-1) = n^2 + (2(n+1)-1)$$

- Use algebra on r.h.s. to show it is $f(n+1)$

$$\sum_{i=1}^{n+1} (2i-1) = n^2 + 2n + 1 = (n+1)^2 \quad //$$

we have proven inductive step $P(n) \rightarrow P(n+1)$

still need the basis step:

$P(1) = (1)^2 = 1$ ✓

* Sums are easy! well defined methodology just shown.

Proof by induction for inequalities.

For example: $P(n) := n < 2^n$ for all $n \in \mathbb{Z}^+$

- Inductive hypothesis, $P(n)$
- Inductive proof step, $P(n) \rightarrow P(n+1)$
- Bases step $P(n_{min}) = T$

- Hypothesis $P(n) := n < \cancel{n} 2^n \; \forall n \in \mathbb{Z}^+$

- Inductive proof step

$P(n)$: $n < 2^n$

Wrote l.h.s of $P(n+1)$

$$\left.\begin{aligned} n+1 &< (2^n)+1 \\ &< 2 \cdot 2^n \end{aligned}\right\} \text{— by Lemma 1}$$

$$= 2^{n+1}$$

$$\therefore n+1 < 2^{n+1}$$

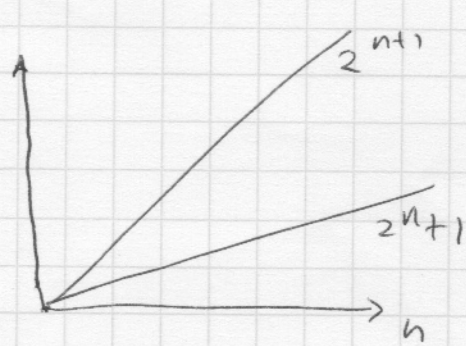
$$(P(n) \rightarrow P(n+1)).$$

verify

n	2^n+1	$2 \cdot 2^n$
1	3	4
2	5	8
3	9	16
⋮		

Lemma 1:

For $n > 0$, $2^n + 1 < 2^{n+1}$



Proof by induction on sets

For example:

$P(n) :=$ A set with n elements has 2^n subsets.

Want to show (1) $P(n) \to P(n+1)$; (2) $P(n_{min}) = T$

1) $P(n) \to P(n+1)$

Assume $P(n)$ a set of size n has 2^n subsets

Let $|S| = n$

Let $|B| = n+1$ and $B = S \cup \{b\}$ where $b \notin S$

For every subset of S, can make a new subset w/ $\{b\}$ included.

$$\# \text{ subsets of } B = 2 \# \text{ of subsets of } S$$

$$= 2 \cdot 2^n = 2^{n+1}.$$

$P(n) :=$ # of subsets of S is 2^n.

$\therefore P(n) \to P(n+1)$

2) Bases step.

$P(n)$ holds for some n_{min}

$P(n=0)$; $|\{\}| = 0$, # of subsets $\{\} = 2^0 = 1$.

$P(n=0)$ is true (that a set with zero elements has 2^0 subsets).

More proofs for sets:

Hypothesis:

$$P(n) := (A_1 \cup A_2 \cup \cdots A_n) \cap B = (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots (A_n \cap B)$$

Basis step: try $n = 1$.

$$P(1) : \quad A_1 \cap B = A_1 \cap B \quad //$$

Inductive step: prove $P(n) \to P(n+1)$

Use distributive law:

$$(X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z)$$

Show $P(n) \to P(n+1)$

$$P(n+1) = (\underbrace{(A_1 \cup A_2 \cup \cdots \cup A_n)}_{X} \cup \underbrace{A_{n+1}}_{Y}) \cap \underbrace{B}_{Z}$$

$\overset{\text{dist law}}{=}$

$$\underbrace{(A_1 \cup A_2 \cup \cdots \cup A_n) \cap B}_{P(n)} \cup (A_{n+1} \cap B)$$

$$= (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots (A_n \cap B) \cup (A_{n+1} \cap B)$$

$$= P(n+1) \quad //$$

Recap proof by induction

- Hypothesis $P(n)$ for $n \in \mathbb{N}, \mathbb{Z}^+$
- Inductive step $P(n) \to P(n+1)$

  (using def'n of $P(n)$ and algebra)
- Basis step $P(n) = T$ for some n_{min}

$$\therefore \forall_{n \geq n_{min}} P(n)$$

Game plan for $P(n) \to P(n+1)$:

- Write the l.h.s. of $P(n+1)$, express it in terms of $P(n)$
- Substitute the expression for $P(n)$ with the hypothesis
- Show this results in appropriate formula

  for r.h.s. of $P(n+1)$

New topic: **Recursion and iteration**

build from top down. build up from bottom

Recursion: a general term for defining an object in terms of itself.

- An inductive proof establishes truth of $P(k+1)$ due to truth of $P(k)$.

- A recursive defn of a func, predicate, set, etc. defines larger elements in terms of smaller elements.

For instance consider the geometric sequence

$\{a_0, a_1, a_2, \dots a_m\}$ where $a_n = ar^n$

(for a, n consts)

A recursive defn of geometric series:

$$\boxed{a_n = (a_{n-1})r}$$

For instance; arithmetic series:

$\{a_0, a_1, \dots a_m\}$ where $a_n = a_0 + nd$

for a_0, d constants.

Recursive defn.

$$\boxed{a_n = a_{n-1} + d}$$

A recursive fun'c $f(n)$ is defined in terms of smaller values of the fun'c.

e.g. $f(0), f(1), f(2) \dots f(k)$ given

For $n > k$ there is a rule for writing

$f(n)$ in $f(0), f(1), f(2), \dots, f(n-1)$.

Example recursive fun'c:

Explicit: Factorial func' $f(n) = n \cdot (n-1)(n-2) \cdots 2 \cdot 1$

Recursive factorial fun'c:

$f(0) = 1$ (Base case)

$\forall_{n \geq 1} f(n) = n \cdot f(n-1)$ Recursive defn.

Explicit fun'c $f(n) = a^n$

Recursive defn: $f(0) = 1$

$\forall n \geq 1 \quad f(n) = a \cdot f(n-1)$