Lec 11: Mid term review

- Review session Weds 5-6:30 in kiebler #3.

Propositions, p, q, r, etc $\xrightarrow[\text{map to}]{} \{T, F\} = \{1, 0\}$

- Compound prop's: operators $\neg, \wedge, \vee, \oplus$
- Truth tables; length of table $= 2^{\#\text{ of propositions}}$

  (easier than symbolic logic derivations)

- Implication $p \to q$ or $P(x) \to Q(x)$

  Only false / invalid if $q = F$ and $p = T$

  $$(p \to q) \equiv \neg p \vee q$$

  - contrapositive $\neg q \to \neg p \equiv p \to q$

  language: $p \to q$
  - p is sufficient for q
  - if p then q
  - q if p
  - q is necessary for p
  - p only if q

- Bi-implication $(p \to q) \wedge (q \to p) =: p \leftrightarrow q$

  "if and only if"
  "necessary and sufficient"

Logical equivalences (symbolic logic)

* See the tables on canvas $\underbrace{\text{Tables 6,7,8}}_{\text{Sec 1.3}}$, $\underbrace{\text{Table 1}}_{\text{Sec 1.6.}}$

De Morgan's:

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

(generalizes to multiple propositions
$\neg(p \vee q \vee r \vee z \dots)$)

· Predicate logic

$P(x)$ ← subject, $P(x,y)$, $Q(x,y,z)$, etc.
predicate

Once subject is given $(P(x)) \xrightarrow[\text{to}]{\text{maps}}$ Proposition $\xrightarrow[\text{to}]{\text{maps}} \{T, F\}$

· Universal and existential quantification
(i.e. for which elements $x \in U$ is $P(x) = T$?)

$\forall x$ "for all x"
$\exists x$ "there exists an x"

can specify the domain U, e.g. $\forall x \in \mathbb{R}$, $\exists x \in \mathbb{Z}^+$, etc.

De Morgan's
$$\neg \forall_x P(x) \equiv \exists_x \neg P(x)$$

$$\neg \exists_x P(x) \equiv \forall_x \neg P(x)$$

Build logical arguments w/ rules of inference.

statement 1
statement 2
⋮

$\left.\right\}$ assumptions and reasoning

$\therefore$ conclusion

Rules of inference

- modus ponens (affirms)
- modus tonens (denies)
- Hypothetical syllogism

$$(p \to q) \land (q \to r) \to (p \to r)$$

- Disjunctive syllogism (useful for simplifying compound props).

$$((p \lor q) \land \neg p) \to q$$

Types of "trivial" proofs

$(p \to q) \land \neg p$ then "vacuous" proof.

$(p \to q) \land q$ then "trivial" proof.

p	q	$p \to q$
1	0	F
1	1	T
0	0	T
0	1	T

Fallacies:

Affirming the conclusion
saying $(p \to q) \wedge q$ therefore p

Denying the antecedent / premise
saying $(p \to q) \wedge \neg p$ therefore $\neg q$

Sets select elements $x \in \mathcal{U}$.

Set builder notation

$A = \{x \mid P(x)\}$ (all $x \in \mathcal{U}$ for which $P(x) = T$)

$B = \{x, y \mid \underbrace{P(x, y)}\}$

* can be a complex instance (e.g. compound prop, implication, etc)

Equivalence of sets

$A = B \leftrightarrow \forall x \{x \in A \leftrightarrow x \in B\}$

Remember in a set each element occurs at most once; and the order does not matter.

Cardinality of a set $|A| \equiv$ # of elements.

Empty set $\phi = \{\}$

Empty set is a valid subset of every set.
(but not a member of every set)

Subsets; $A \subset C$ all $a \in A$ are also $a \in C$. but there is an $c \in C$ with $c \notin A$

$A \subseteq C$ if ~~A~~ A can equal C.

subset is all ways of choosing a set of elements from a given set.

Power set, $P(A)$ = set of all possible subsets of A.

$$|P(A)| = 2^{|A|}$$

e.g. $A = \{1, 2, 3\}$ or $B = \{1, \{1,2\}, 3\}$

an element of a set can be a set.

$|B| = 3$, $|P(B)| = 2^3 = 8$

$$P(B) = \{\{\emptyset\}, \{1\}, \{\{1,2\}\}, \{3\}, \{1, \{1,2\}\}, \{1, 3\}, \{\{1,2\}, 3\}, \{1, \{1,2\}, 3\}\}$$

- Venn diagrams
- Cartesian Products
- De Morgan's $\overline{A \cup B} = \overline{A} \cap \overline{B}$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

- Union, intersection and substraction of sets.

- Union of a collecton of sets $\bigcup_{i=1}^{N} A_i$

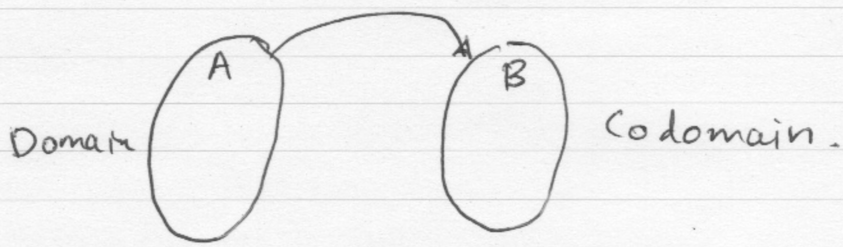
- Intersection " " " "

$$\bigcap_{i=1}^{N} A_i$$

<u>Functions</u> $f: A \rightarrow B$ (maps A to B)

Is a specific mapping:

· A function? – need every $a \in A$ is mapped to one (not necessarily unique) $b \in B$.



- 1-to-1 (injective); a <u>function</u> where every $a \in A$ maps to a unique $b \in B$.
  (each $b \in B$ has at most one partner in a)
  $|\text{domain}| \leq |\text{codomain}|$

- onto (surjective); a function where every $b \in B$ has at least one partner in A.
  $|\text{domain}| \geq |\text{codomain}| = |\text{range}|$

· Bijection is both 1-to-1 and onto.

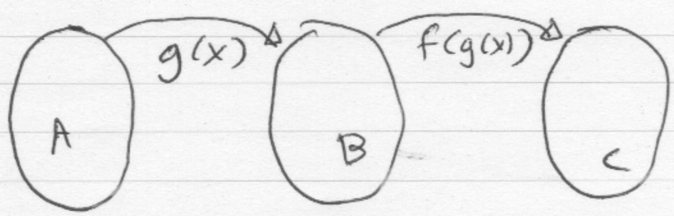
Every $a \in A$ has a unique partner in B.
Every $b \in B$ " " " " " A.

$|\text{domain}| = |\text{codomain}| = |\text{range}|$

* A bijection means $f^{-1}(x)$ exists.

Func. composition:

$(f \circ g)x = f(g(x))$



Inverse func composition $(f \circ f^{-1})x = x$.

* Choice of domain & codomain can make all the difference

Series : an order list, where repitition matters

Summations of sequences.

$$\sum_{k=0}^{n} a_k = a_0 + \sum_{k=1}^{n-1} a_k + a_n \quad \text{, etc.}$$

$$= a_0 + \sum_{k=1}^{j} a_j + \sum_{l=j+1}^{n} a_l$$

Imp. sums

$$\sum_{j=1}^{r} j \quad , \quad \sum_{j=1}^{n} j^2 \quad , \quad \text{geometric series,}$$

arithmetic series.

Proofs

Need some number theory background

- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{R}$, etc.
- Properties of these sets.
  - rational versus irrational
  - odd versus even
  - positive, negative, zero
  - prime versus composite.

(Needed these definitions as our axioms).

Methods of proof of an implication

- Direct proof $p \to q$
- Contrapositive $\neg q \to \neg p$
- Proof by contradiction
    - A proposition P; show $\neg P \to (r \wedge \neg r)$
      (where $P \to r$, typically a mathematical defn of ~~property~~ proposition P).
    - of implication $p \to q$

      Show $(p \wedge \neg q) \to (p \wedge \neg p) \vee (q \wedge \neg q)$

      Argument

$$\begin{array}{ll} p & \text{assert} \\ \neg q & \text{assert} \\ \hline \therefore (p \wedge \neg p) \vee (q \wedge \neg q) & \end{array}$$

- Proof by cases
- Constructive existence proofs
- Nonconstructive existence proofs.