1. **Direct proof.**
   The implication $p \to q$ can be proved by showing that if p is true then q must also be true. A proof of this kind is called a *direct proof*.

2. **Indirect Proof.**
   Proof by contraposition: Since the implication $p \to q$ is equivalent to its contrapositive, $\neg q \to \neg p$, the implication $p \to q$ can be proved by showing that $\neg q \to \neg p$ is true. This related implication is usually proved directly. An argument of this type is called an *indirect proof*.
   Example: Prove "If $3n + 2$ is odd, then n is odd".

   A **vacuous proof** is established by showing $\neg p$.
   A **trivial proof** is established by showing q is true.

3. **Proof by contradiction.**
   (a) *For proposition* p: Assume $\neg p$ is true and show this leads to both r and $\neg r$ for some independent proposition r; in other words $\neg p \to (r \wedge \neg r)$.

   (b) *For implication* $p \to q$: By assuming that the hypothesis p is true and that the conclusion q is false, then using p and $\neg q$ as well as other axioms, definitions, and previously derived theorems, derives a contradiction. Proofs are based on noting that

   $$(((p \to q) \wedge p) \wedge \neg q) \equiv (q \wedge \neg q) \quad \text{likewise} \quad (p \wedge (\neg q \to \neg p)) \equiv (p \wedge \neg p).$$

   Exmpl (a): Prove that $\sqrt{2}$ is irrational.
   Exmpl (b): Prove that for all real numbers x and y, if $x + y \geq 2$, then either $x \geq 1$ or $y \geq 1$.

4. **Equivalence proof** (or "if-and-only-if proof", "necessary-and-sufficient proof").
   To prove a theorem that is an equivalence, i.e., $p \leftrightarrow q$, the tautology

   $$(p \leftrightarrow q) \leftrightarrow ((p \to q) \wedge (q \to p))$$

   can be used. That is, the proposition "p if and only if q" can be proved if both the implication "if p then q" and "if q then p" are proved.
   Example: Prove the theorem: The integer n is odd if and only if n^2 is odd".

5. **Exhaustive proof / proof by cases**

   Exhaustive proof: Proof by showing it holds for all possible x in U (e.g. a truth table).

   Proof by cases: Proof by showing it holds for all possible cases. (Useful when direct proof not simple but the extra information in the cases let's you move forward.)

   Example: Prove that if n is an integer, then $n^2 \geq n$. (Three cases, $n < 0, n = 0, n > 0$.)

   Without loss of generality (WLOG): Same proof holds for all cases. Quite useful but gets one into trouble (a common mistake in a proof).

6. **Constructive existence proof:** of statement of the form $\exists x P(x)$. Just find one value of x in U for which $P(x)$ is true. (Hence "constructive")

   Example: Prove there exists a positive integer n that is the sum of cubes of positive integers in two ways: $\exists n : n = i^3 + j^3 = k^3 + l^3$.

7. **Nonconstructive existence proof.** Don't pinpoint the exact values that satisfy, just show they must exist.

   Example: Show there exist irrational numbers x and y such that x^y is rational. In propositional language $\exists x, y$ ((Irrational(x) $\wedge$ Irrational(y)) $\rightarrow$ Rational(x^y)).

   Example: Prove there are infinitely many primes; $\forall n \exists p > n$ where p is prime.

8. **Proof by counterexample.**

   The goal of such a proof is to show $\forall x P(x)$ is false.

   (Note: Showing $\exists x P(x)$ is false is counterexample for $\forall x P(x)$, but this is not a counterexample for the conjecture $\exists x P(x)$.)

   Example: Show that the assertion "All primes are odd" is false.

9. **Mistakes in proofs.**

   False premises

   Circular reasoning

   Not considering all cases (e.g. missing $x = 0$ case).

   Showing $\exists$ a satisfying value to prove $\forall$ statements.

   Assuming $\exists$ means that all instances satisfy.

   Example: What is wrong with the famous supposed "proof" that $1 = 2$?