**ECS 253 / MAE 253, Network Theory and Applications**
Spring 2018
**Common Problem Set # 2, Due May 3**

## Problem 1: Finite size scaling.

Consider a network where node degree $k$ follows a power law degree distribution $p_k = (\gamma - 1)k^{-\gamma}$, with $\gamma > 1$. We will approximate $k$ as continuous and then let $P_K$ denote the cumulative distribution function (CDF) which is the probability a node will have degree less than or equal to $K$,

$$P_K = \int_1^K p_k dk.$$

Here we will work out an estimate for the maximum node degree, $K_{\max}$, that one would expect to see in a network of size $N$ with a power law degree distribution. Operationally, we define the expected value of $K_{\max}$ for a network of size $N$ to be the value of degree when we expect only one node bigger than this value:

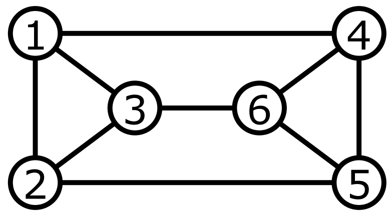
$$N(1 - P_{K_{\max}}) \approx 1.$$

Using this show that $K_{\max} \approx N^{1/(\gamma-1)}$ and evaluate the expression explicitly for $\gamma = 2, 3, 4$.

## Problem 2: Modularity matrix

You would probably want to use a software package in order to compute the eigenvalues and eigenvectors for the matrices involved in this problem. Note that the methods shown below will work only for splitting the network into two parts. You cannot use the same method to further split the network.

**2.1) Bisection of a binary undirected network**




1. Consider the graph given above. Give the adjacency matrix ($\mathbf{A}$) of this graph.

2. Let $m$ denote the total number of edges in this graph. Construct a matrix $\mathbf{B}$ such that

$$B_{ij} = A_{ij} - \frac{k_i k_j}{(2m)}. \tag{1}$$

If we define a column vector $\mathbf{k}$ such that the $i$th component is the degree of node $i$, we can express $\vec{B}$ as

$$\mathbf{B} = \mathbf{A} - \frac{\mathbf{k} * \mathbf{k}^T}{2m}. \tag{2}$$

Note that $\mathbf{B}$ is called the modularity matrix.

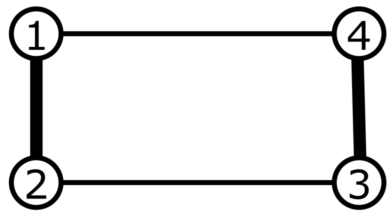
3. Obtain the eigenvalues of $\mathbf{B}$ and list them.

4. What is the eigenvector (say $\vec{v}$) corresponding to the largest eigenvalue?

5. Look at the sign of each component of $\vec{v}$. If $v_i > 0$ assign node $i$ to community 1 else assign node $i$ to community 2. List the nodes that belong to community 1 and nodes that belong to community 2. Is the assignment reasonable?

6. Why does this work?

**2.2) Bisection of a weighted undirected network** For this section we will use an approach very similar to the one described above but slightly modify it to handle a weighted network. In the graph below the thicker edges have a weight of 5 and the other edges have a weight of 1.




1. Consider the graph given above. Give the weighted adjacency matrix ($\mathbf{A}$) of this graph.

2. Let $m$ denote the total weight of edges in this graph. You can obtain $m$ by summing all the entries in $\mathbf{A}$ and dividing it by 2.

3. Construct a matrix $\mathbf{B}$ such that

$$B_{ij} = A_{ij} - \frac{k_i k_j}{(2m)}. \tag{3}$$

   If we define a column vector $\mathbf{k}$ such that the $i$th component is the total weight associated with node $i$ (i.e, sum the weight of all edges that involve node $i$), we can express $\mathbf{B}$ as

$$\mathbf{B} = \mathbf{A} - \frac{\mathbf{k} * \mathbf{k}^T}{2m}. \tag{4}$$

4. Obtain the eigenvalues of $\mathbf{B}$ and list them.

5. What is the eigenvector (say $\vec{v}$) corresponding to the largest eigenvalue?

6. Look at the sign of each component of $\vec{v}$. If $v_i > 0$ assign node $i$ to community 1 else assign node $i$ to community 2. List the nodes that belong to each of the two communities? Is the assignment reasonable?