Phase Transitions in Networks: Giant Components, Dynamic Networks, Combinatoric Solvability

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Outline

Traditional Thermodynamics

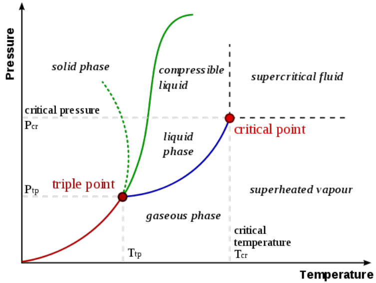
- Classical phases
- Classical transitions
- 

Figure: Graciously stolen from Wikipedia.org

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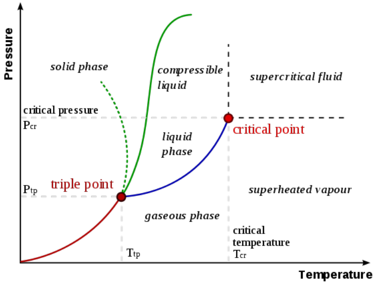

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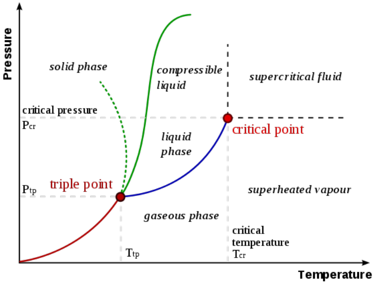

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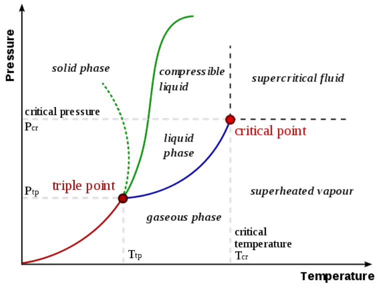
- 

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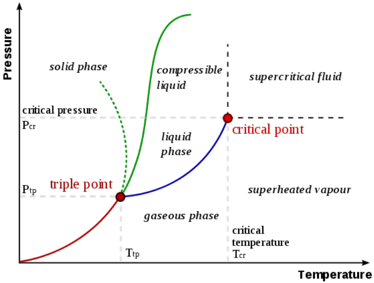
- 

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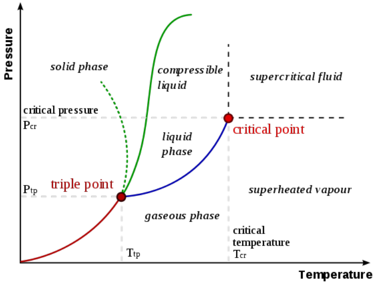
- 

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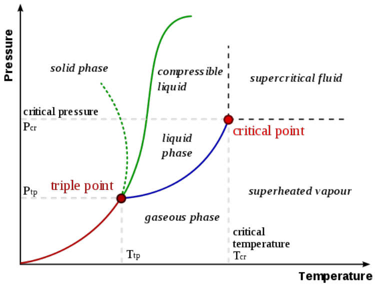
- 

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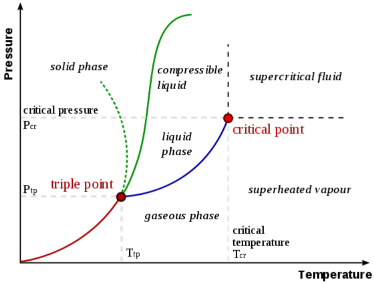
- 

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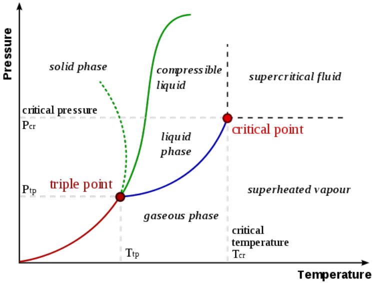
- 

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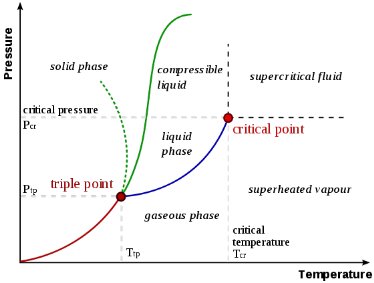
- 

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  - Scaling Exponents
  - Renormalization
  - Universality

Ising Model: System Description

The Ising model is one of the most extensively studied models in physics. It consists of a lattice with a spin located at each site. These spins, S_i, take a value of $\pm\frac{1}{2}$.

$$E = -\sum_{<i,j>} J_{ij} S_i S_j = -J \sum_{<i,j>} S_i S_j$$

$$S_i \in \{+\frac{1}{2}, -\frac{1}{2}\}$$

Ising Model: Phase Transition

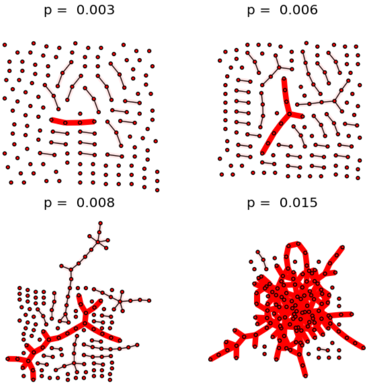
At temperatures $T > T_c$ we see that the magnetization (average spin of the lattice) $m = 0$. However, for $T < T_c$, $m \neq 0$.
The most interesting behavior occurs when $T \approx T_c$. In this regime things scale according to power laws.

Renormalization & Universality

- Renormalization
  - Scale-free
  - Power laws
- Universality
  - The Ising critical point
  - Bethe Lattice / Cayley Tree percolation
  - H_2O critical point

Erdös-Réyni Random Graphs

Random graphs were shown to exhibit phase transitions, from a phase with a slowly growing ($N^{\frac{2}{3}}$) largest connected component to a phase where the largest connected component (known now as the giant component) is faster growing (cN).

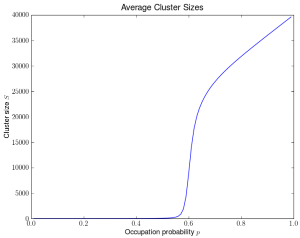


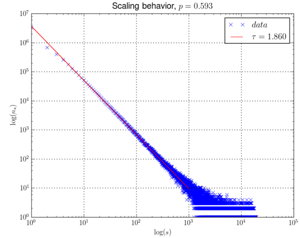
Outline

1. Historical Prospective
2. Dynamic Percolation
   - Traditional Percolation
   - Dynamic Percolation
3. Combinatoric Problems
4. Conclusions

Static Percolation

- Start with an empty lattice
- Populate the sites with probability p
- If $p < p_c$, only clusters of finite size exist
- If $p > p_c$, an infinite or percolating cluster exists

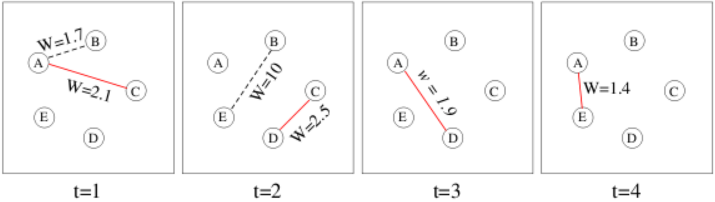




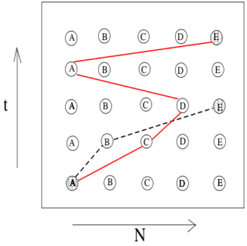
Definitions

- A network consists of N nodes and M edges
- Each edge has a lifetime τ drawn from a Poisson distribution
- Once an edge's lifetime has expired, it is removed from the network and a new edge is placed between two nodes at random
- A 'walker' takes unit time to transverse an edge, and may only linger at a node for a limited time

Dynamics

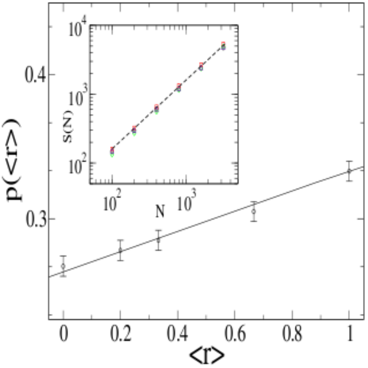


Two paths through the dynamic network. The red path, though longer, has lower weight than the dashed path.



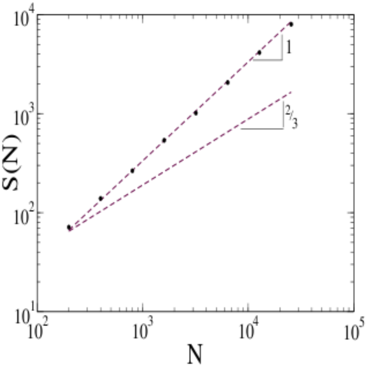
p_c Dependency & Critical Behavior

- The value of p_c depends on the value of $< r >$, the average rate at which edges turn over
- The size of the giant component, however, does not depend on p_c
- Thus the critical behavior is independant of $< r >$



Differences from Static

- In a static network, the size of the giant cluster scales as $N^{\frac{2}{3}}$
- It is found that in the dynamic network, the giant cluster scales as N
- This is due to the dynamic network having effectively $N^{\frac{3}{2}}$
- This is because the temporal axis has a depth of $N^{\frac{1}{2}}$, making the full number of states $NN^{\frac{1}{2}} = N^{\frac{3}{2}}$
- ...and thus $(N^{\frac{3}{2}})^{\frac{2}{3}} = N$



Universality Class

And so, since the critical exponents for the dynamic network are different than the static network, they belong to different universality classes!

Outline

Definition

K-SAT:

- Given the logical AND of M clauses...
- Each consisting of logical OR of K boolean varialbles...
- Drawn from a total set of N boolean variables...
- Is there an assignment of values to those N for which each of the M clauses evaluates to True?

Example:

$$(x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_3)$$

is satisfied by

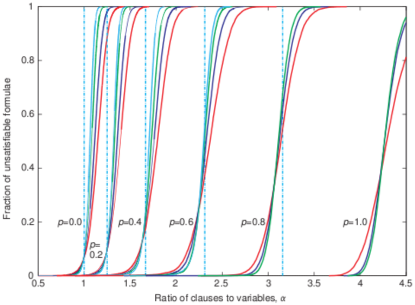
$$(x_1, x_2, x_3) = (\top, \top, \top)$$

How difficult is K-SAT?

K-SAT ($K \geq 3$) belongs to a complexity class called NP-Complete. That is, it can be solved by a non-deterministic Turing machine is time polynomial in the size of the input. Put another way, if someone gives you a possible solution you can test it in polynomial time (linear in this case).
But the question arises – is K-SAT *always* hard?

Sometimes K-SAT is easy!

- If the ratio of clauses to variables is low, it's easy to find a solution as there are very few restrictions
- If the ratio of clauses to variables is high, it's very unlikely that there is a solution as the variables are over-constrained
- And somewhere in the middle (depending on K) there is a very sharp transition, and this is where K-SAT is very difficult



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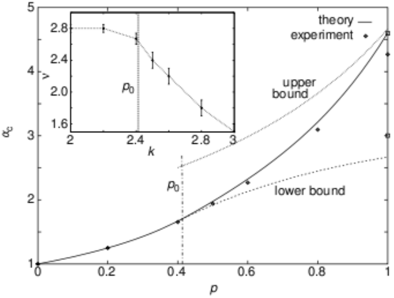
  - $m_i \approx 0$ for unconstrained variables

Phase Transitions in the Ising-like Model

- Define $f(K, \alpha)$ to be the fraction of constrained variables for an equation with clause-size K and $\alpha = M/N$



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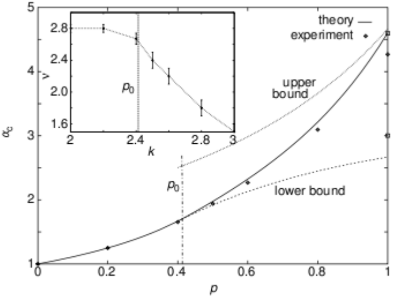

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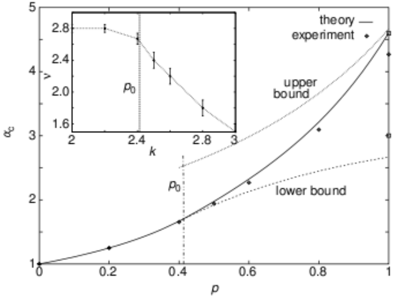

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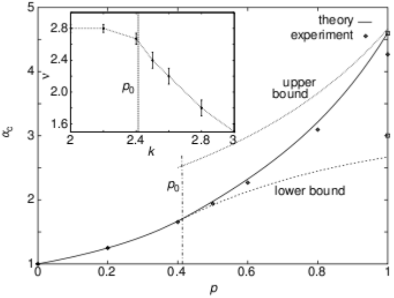
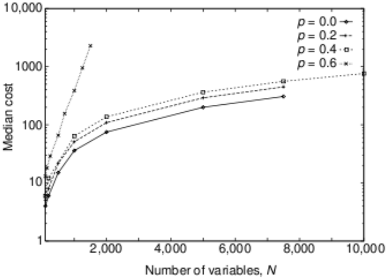
- The transition from continuous to discontinuous phase transition occurs at $K_c \approx 2.41$



Computational Cost in K-SAT



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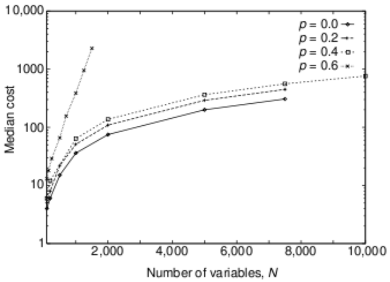

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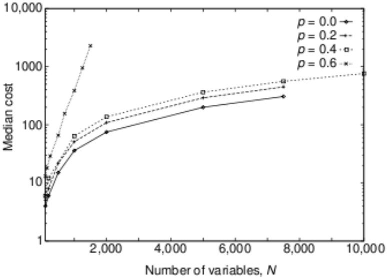
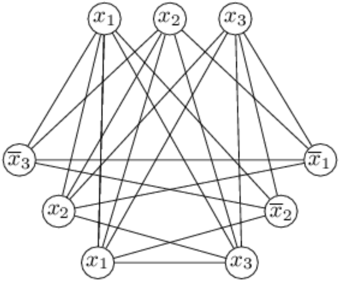
- However for $K > K_c$ the cost grows exponentially, which is consistent with 3-SAT being in NP



Another Mapping: K-SAT to CLIQUE



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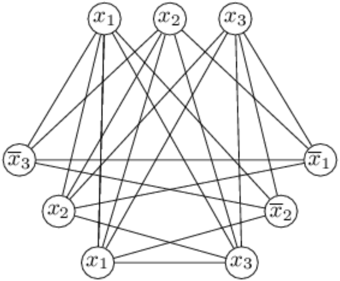

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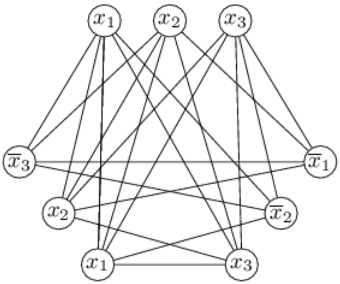

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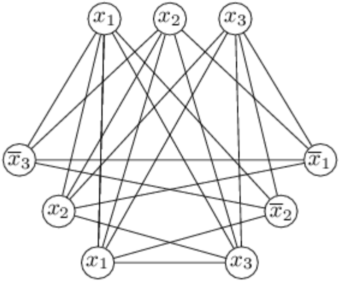

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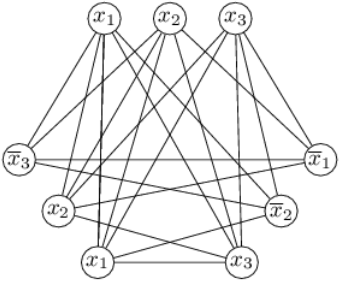

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- Write each clause in a little group
- Connect each node to any node that isn't its negation and also isn't in its own group
- If there is an M-CLIQUE, then this equation is satisfiable

Outline

Conclusions

- Phase transitions tend to occur at the most interesting places in parameter space
- They correspond to qualitatively different behavior
- Studying the phase transition can give us valuable insight into how to solve problems, what problems can be solved, and other insights regarding problem spaces with distinct phases