Synchronization and Pattern Evolution on Networks: The Interplay of Structure and Dynamics

Ben Johnson

Graduate Group in Applied Math

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Dynamics on Networks

What does this mean?

- Focus on processes (diffusion, synchronization, proliferation) occurring on networks
- Functionality and efficiency of such processes relative to network topology and dynamics

Why do we care?

- Understand real-world networks
- Find a connection between structure and function

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Solution: Narrow our focus

- Synchronization
- Pattern evolution

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What is synchronization?

- Intuitive answer: Highly similar behavior

Can we be more precise?

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- Generalized synchronization
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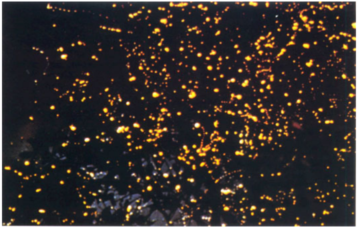
Importance of Synchronization

- Synchronization is observed in many real-world networks
  - Fireflies flashing together
  - Neurons firing in a neural network
  - Heart pacemaker cells
  - Coupled laser arrays
- Understanding these may shed light on other networks
  - Connection to network structure?
  - Reveal unseen function



Kuramoto Model

Synchronization itself is very broad, simplify analysis by using the Kuramoto model:

- Established, standard model for synchronization
- Well studied
- Simple, yet robust

Kuramoto Model

Given N coupled oscillators, whose dynamics satisfy

$$\frac{d\phi_i}{dt} = \omega_i + \sum_{j=1}^{N} J_{ij} \sin(\phi_j - \phi_i) + I_{i,m}$$

- $\phi_i(t)$ = phase of oscillator i at time t
- ω_i = natural frequency of oscillator i
- J_{ij} = coupling strength between oscillators i and j
- $I_{i,m}$ = external driving strength to oscillator i for driving condition m

Perturbations Near Synchronization

Consider the difference between the perturbed and unperturbed system

$$
\begin{aligned}
D_{i,m} &= \Omega_m - \Omega_0 - I_{i,m} \\
&= \sum_{j=1}^{N} J_{ij}[\sin(\phi_{j,m} - \phi_{i,m}) - \sin(\phi_{j,0} - \phi_{i,0})] \\
&\approx \sum_{j=1}^{N} L_{ij}\theta_{j,m}
\end{aligned}
$$

- L is the Laplacian matrix
- Ω_m and Ω_0 are the driven and undriven collective frequencies
- $\theta_{j,m} = \phi_{j,m} - \phi_{j,0}$

Results

- Each driving condition m yields $N - 1$ independent phase shifts ($\theta_{i,m}$) and a collective frequency Ω_m
- Gives N of possible N^2 network connections
- M driving conditions provide MN restrictions $\Rightarrow$ need at most N experimental runs
- Reveals strength of connection

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Difficulties

- Difficult to solve $D = L\theta$ (ill-conditioned)
- Network size
- Cost of each experiment

How can we improve this method?

Improvement

- Realize that most networks do not have N^2 connections
- Use singular value decomposition to create the matrix $\hat{J}$ and minimize $||\hat{J}||_1$
- Result: sparsest matrix that satisfies the system equations (minimal connections)

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Quality of reconstruction

Element-wise difference between real and computed connectivity matrices:

$$\Delta J_{ij} : \frac{|J_{ij}^{\text{derived}} - J_{ij}^{\text{actual}}|}{2J_{\max}}$$

Quality of reconstruction to accuracy α after M experiments:

$$Q_\alpha(M) := \frac{1}{N^2} \sum_{i,j} H((1 - \alpha) - \Delta J_{ij}),$$

where H is the Heaviside step function ($H(x) = 1$ for $x \geq 0$).

Quality of Reconstruction

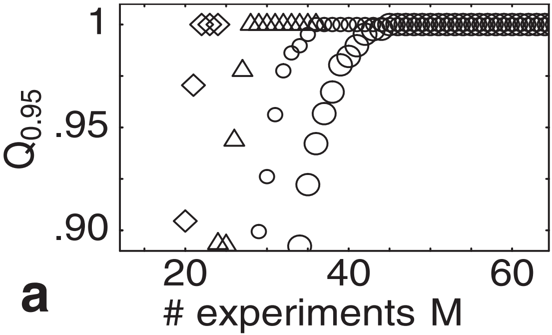

Figure: Quality of reconstruction and required number of experiments. Quality of reconstruction ($\alpha = .95$) for $k = 10$ and $N = 24(\diamond)$, $N = 36(\triangle)$, $N = 66(\circ)$, and $N = 96(\bigcirc)$

Minimum Number of Experiments

Minimum number of experiments for accurate reconstruction on quality level q:

$$M_{q,\alpha} := \min\{M : Q_\alpha(M) \geq q\}$$

- Assuming $0 < 1 - \alpha \ll 1$ and $0 < 1 - q \ll 1$
- Sublinear in numerical experiments
- Connectivity can be determined even if $M \ll N$

Minimum Number of Experiments

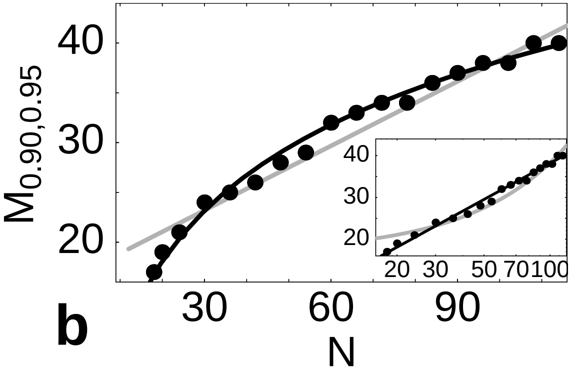

Figure: Minimum number of experiments required ($q = .90, \alpha = .95$) versus network size N with best linear and logarithmic fits (gray and black solid lines). Inset show same data with N on logarithmic scale.

Community detection

Synchronization dynamics can reveal the connectivity of a network

Very often, we wish to know more than just connectivity. Can we detect community structure as well?

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- Start with Kuramoto model for coupled oscillators:

$$\frac{d\phi_i}{dt} = \omega_i + \sum_{j=1}^{N} J_{ij} \sin(\phi_j - \phi_i) + I_{i,m}$$

  With $I_{i,m} = 0$ (undriven network)
- Look at average correlation between pairs of nodes. Define local order parameter:

$$\rho_{ij}(t) = \langle \cos(\phi_i(t) - \phi_j(t)) \rangle$$

- Why cosine?

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Dynamic Connectivity Matrix

Convert correlation matrix $[\rho_{ij}]$ into a binary matrix.

Define

$$D_t(T)_{ij} = \begin{cases} 1 & \text{if } \rho_{ij}(t) \geq T \\ 0 & \text{if } \rho_{ij}(t) < T \end{cases}$$

T is some threshold value.

- Different values of T reveal different levels of structure in the network
- Fix a threshold T and look at time evolution

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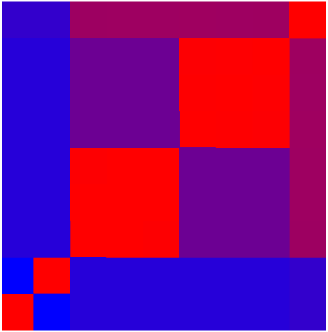
Visualization of Dynamic Connectivity

What are the communities of this network?



Red for shorter times, blue for longer times

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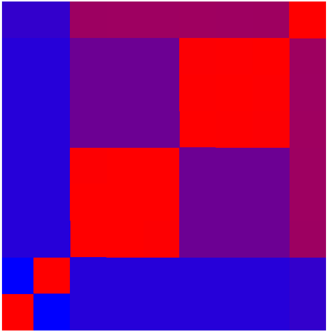

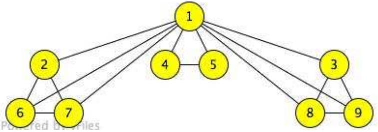

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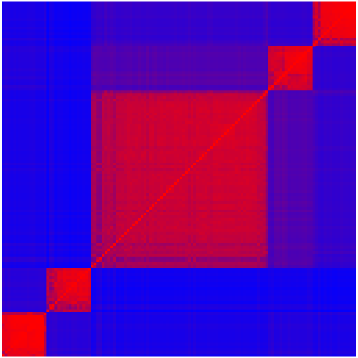
Examples

What about this network?



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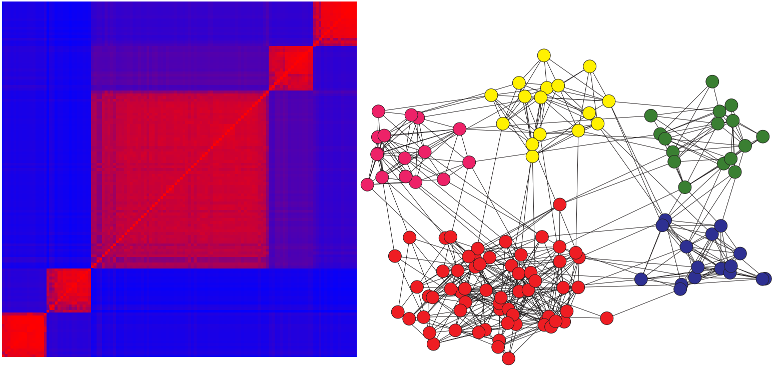

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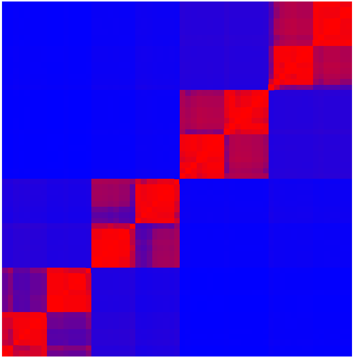

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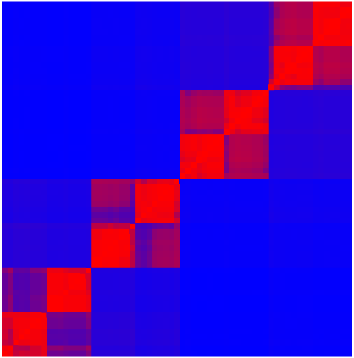

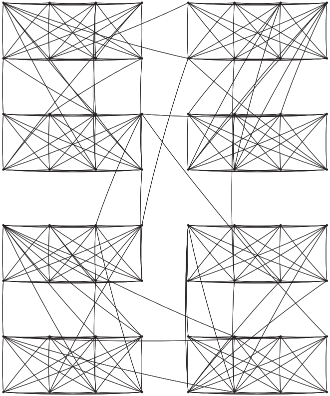

Results

- Accurately detects the community structure of a network
- Also detects substructure within communities
- Reveals equivalence between disconnected communities

Pattern Evolution

Dynamics can reveal a lot of information about network connectivity and community structure

Can network structure predict the behavior of the dynamics?

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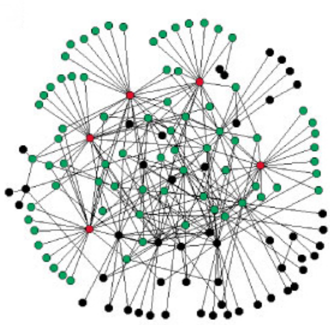
Scale-Free Networks

Recall that our degree distribution follows a power law:

$$P(k) \sim k^{-\gamma}$$

For our purposes (and in many real-world networks) $2 < \gamma < 3$



Our Model

- Undirected network with scale-free degree distribution
- Vertex degree governed by $k_0 \leq k \leq k_{\text{max}}$ with $k_0 \geq 2$ and $k_{\text{max}} \sim N^{1/\gamma - 1}$
- Average vertex degree $\langle k \rangle \geq 10$
- Each vertex has a binary, Ising-like spin variable

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Time evolution

We use local majority dynamics

- State of vertex i at time t is $\sigma_i(t) = \pm 1$.
- Evolution of system:

$$\sigma_i(t+1) = \begin{cases} +1 & \text{if } h_i(t) > 0 \\ -1 & \text{if } h_i(t) < 0 \\ \pm 1 \text{ with } \mathbb{P} = \frac{1}{2} & \text{if } h_i(t) = 0 \end{cases}$$

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To study evolution patterns, consider

- $q_k(t) =$ probability that a vertex of degree k is +1
- $Q(t)$= probability that for any vertex chosen, a random neighbor is +1

A vertex associated with a random edge has degree = k with probability $\frac{kP(k)}{\sum_k kP(k)} = \frac{kP(k)}{\langle k \rangle}$.

Then

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Time evolution of model

Given our previous description of local majority dynamics, we see

$$q_k(t+1) = \sum_{m=\lceil k/2 \rceil}^{k} \left[1 - \frac{1}{2}\delta_{m,k/2}\right] \binom{k}{m} Q^m(t)[1 - Q(t)]^{k-m}$$

and

$$\Psi(Q) = Q(t+1) = \sum_k \frac{kP(k)}{\langle k \rangle} q_k(t+1)$$

Phase Boundary

It is easy to check that Q has 3 fixed points: $0, \frac{1}{2}$, and 1.

- 0 and 1 are both stable (all + or all - system)
- $\frac{1}{2}$ is unstable phase boundary between attracting fixed points

Define order parameter $y(t) = \left|Q(t) - \frac{1}{2}\right|$

Evolution of order parameter

Working with the equations of our model, we find that

$$y(t+1) \approx \Psi'\left(\frac{1}{2}\right) y(t)$$

$$\Psi'\left(\frac{1}{2}\right) \approx \begin{cases} c_\gamma k_0^{1/2} & \text{for } \gamma > \frac{5}{2} \\ c_\gamma k_0^{1/2} \ln N & \text{for } \gamma = \frac{5}{2} \\ c_\gamma k_0^{1/2} N^{\alpha/2} & \text{for } 2 < \gamma < \frac{5}{2} \end{cases}$$

where $\alpha = \frac{5-2\gamma}{\gamma-1}$

Analytical results

Starting with a *strongly disordered state* ($y(t_0) = \pm\frac{1}{N}$) evolve system using local majority rule dynamics.

Define t_d as the time to reach y^*, that satisfies $|y^*| \geq \frac{1}{4}$
From analysis of our evolution equations, we find $t_d \approx \frac{\ln(\langle k\rangle N)}{\ln(\Psi'(1/2))}$

$$t_d \sim \begin{cases} \ln N & \text{for } \gamma > \frac{5}{2} \\ \frac{\ln N}{\ln(\ln N)} & \text{for } \gamma = \frac{5}{2} \\ 2^{\frac{\gamma-1}{5-2\gamma}} & \text{for } 2 < \gamma < \frac{5}{2} \end{cases}$$

Numerical results

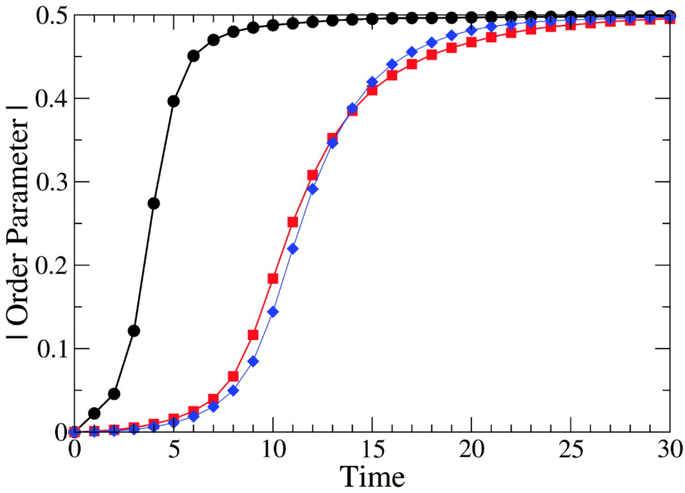
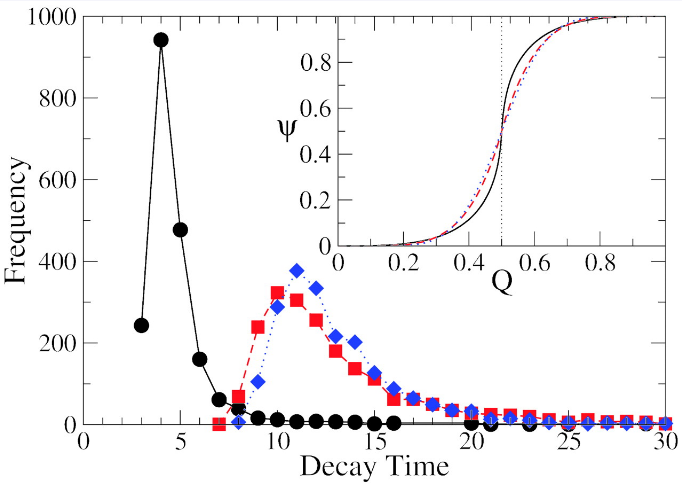


Figure: $\gamma = 2.25$ and $k_0 = 5$ (●), $\gamma = 3$ and $k_0 = 10$ (■), Poissonian network (◆). $N = 2^{18}$, $\langle k \rangle = 20$.



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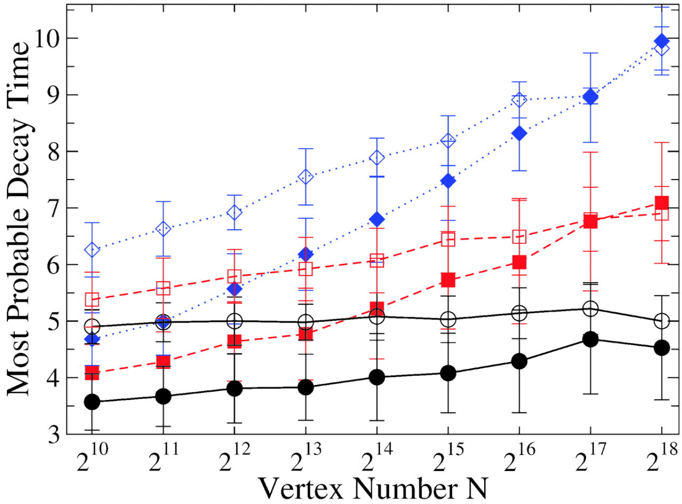

Figure: $\gamma = 2.25$ and $k_0 = 5$ (●), $\gamma = 2.5$ and $k_0 = 7$ (■), $\gamma = 3$ and $k_0 = 10$ (◆), $\langle k \rangle = 20$. Filled = numerical, empty = analytic

Results

- Numerical simulations agree with analysis of evolution equations
- We don't find domains with different patterns (no meta-stability)
- In all numerical runs, the probability of not reaching a completely ordered pattern is less than 10^{-2}
- Decrease in mean vertex degree ($\langle k \rangle$) increases decay time

Changing existing patterns

Given a network in an all-spin-down pattern, how many flips to cause evolution into all-spin-up pattern?

- Simple-minded approach: Choose random vertices - Requires $\sim \frac{N}{2}$ flips
- Better approach: Choose mostly highly connected vertices

Analytic results:

$$\Omega_{\text{min}} \approx 2^{-(\gamma-1)/(\gamma-2)}$$

Note that

$$\lim_{\gamma \to 2^+} \Omega_{\text{min}} = 0 \qquad \text{and} \qquad \lim_{\gamma \to \infty} \Omega_{\text{min}} = \frac{1}{2}$$

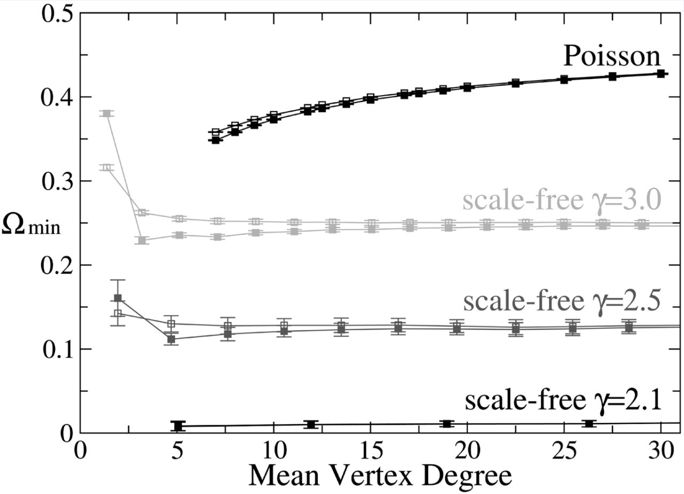

Figure: Minimal fraction $\Omega_{\min}$ of spins that must be flipped to induce transition from all-spin-down to all-spin-up pattern. $N = 10^5$. Open squares = analytic results, Filled squares = numerical results.

Big picture

- $\gamma = \frac{5}{2}$ represents a sharp boundary for pattern evolution on scale-free networks.
- For $2 < \gamma < \frac{5}{2}$ strongly disordered patterns decay in finite even in the limit of large N
- Not the case for $\gamma \geq \frac{5}{2}$

Many real-world networks have $2 < \gamma < \frac{5}{2}$. Why?

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Where to go from here

- Weighted edges in network
- Effect of clustering and modularity
- Dynamic topology
- Interaction delays
- Multi-layered network