ALGORITHMS FOR SPATIAL PYTHAGOREAN-HODOGRAPH CURVES

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Abstract The quaternion representation for spatial Pythagorean–hodograph (PH) curves greatly facilitates the formulation of basic algorithms for their construction and manipulation, such as first–order Hermite interpolation, transformations between coordinate systems, and determination of rotation–minimizing frames. By virtue of their algebraic structures, PH curves offer unique computational advantages over "ordinary" polynomial curves in geometric design, graphics, path planning and motion control, computer vision, and similar applications. We survey some recent advances in theory, algorithms, and applications for spatial PH curves, and present new results on the unique determination of PH curves by the tangent indicatrix, and the use of generalized stereographic projection as a tool to obtain deeper insight into the basic structure and properties of spatial PH curves.

Keywords: Pythagorean–hodograph curves; quaternion representation; rotation–minimizing frame; arc length; elastic energy; Hermite interpolation; tangent indicatrix.

1. Introduction

Pythagorean–hodograph (PH) curves [10, 19] incorporate special algebraic structures that offer computational advantages in diverse application contexts, such as computer aided design, computer graphics, computer vision, robotics, and motion control. The planar PH curves are most conveniently expressed in terms of a complex variable model [7], which helps facilitate key constructions [1, 8, 15, 18, 24, 30]. To achieve a sufficient–and–necessary characterization for spatial PH curves, a quaternion model is required [4, 11]. Our goal in this paper is to survey new algorithms that employ this representation [3, 9, 12–14, 26]; to describe new results on the unique correspondence between PH curves and tangent indicatrices; and to highlight important open problems concerning the theory, construction, and applications of spatial PH curves.

Among the key distinguishing features of any PH curve $\mathbf{r}(t)$ — as distinct from an "ordinary" polynomial curve — we cite the following:

- The cumulative are length $s(t)$ is a *polynomial* in the curve parameter t, and the total arc length can be computed *exactly* (i.e., without numerical quadrature) by rational arithmetic on the curve coefficients [6].

- Integral shape measures, such as the *elastic energy* — the integral of the square of curvature — are amenable to exact closed–form evaluation [8].

- PH curves admit *real–time interpolator algorithms* that allow computer numerical control (CNC) machines to accurately traverse curved paths with speeds dependent upon time, arc length, or curvature [16, 21, 31].

- The *offsets* (or *parallels*) to any planar PH curve admit an exact rational parameterization — likewise for the "tubular" *canal surfaces* that have a given spatial PH curve as the "spine" curve [11, 19, 20].

- An exact derivation of *rotation–minimizing frames* (which eliminate the "unnecessary" rotation of the Frenet frame in the curve normal plane) is possible for spatial PH curves [9] — these involve logarithmic terms; efficient rational approximations are available [13] as an alternative.

- PH curves typically yield "fair" interpolants (with more even curvature distributions) to discrete data — as compared to "ordinary" polynomial splines or Hermite interpolants [1, 8, 18, 20, 30].

Our plan for this paper is as follows. After reviewing basic properties of the quaternion formulation for spatial PH curves in §2, we briefly summarize in §3 the first–order Hermite interpolation problem using this form. Computation of rotation–minimizing frames on spatial PH curves is then discussed in §4. A remarkable property of PH curves is newly identified in §5: they are *uniquely determined* (modulo translation and uniform scaling) *by the tangent indicatrix,* the curve on the unit sphere describing the variation of the tangent vector. In §6 we discuss the *generalized stereographic projection* as a means to obtain deeper insight into the structure of the space of spatial PH curves. Throughout the paper we identify important open problems in the theory, algorithms, and applications of spatial PH curves that deserve further investigation. Finally, §7 summarizes the recent advances and makes some closing remarks.

2. Quaternion formulation of spatial PH curves

The defining characteristic of a Pythagorean–hodograph curve $\mathbf{r}(t)$ in $\mathbb{R}^n$ is the fact that the coordinate components of its derivative or *hodograph* $\mathbf{r}'(t)$ comprise a Pythagorean n–tuple of polynomials — i.e., the sum of their squares coincides with the perfect square of some polynomial $\sigma(t)$. Satisfaction of this condition requires the incorporation of a special algebraic structure in $\mathbf{r}'(t)$, dependent on the dimension n of the space.[1]

The polynomial $\sigma(t)$ defines the *parametric speed* of the curve $\mathbf{r}(t)$ — i.e., the rate of change

$$\sigma = \frac{\mathrm{d}s}{\mathrm{d}t}$$

of its arc length s with respect to the curve parameter t. The fact that $\sigma(t)$ is a *polynomial* (rather than the square–root of a polynomial) in t is the source of the advantageous properties of PH curves.

In the planar case ($n = 2$), a sufficient–and–necessary condition [19] for $\mathbf{r}'(t) = (x'(t), y'(t))$ to be Pythagorean, with $\gcd(x'(t), y'(t)) = \mathrm{constant}$, can be expressed as

$$x'^2(t) + y'^2(t) = \sigma^2(t) \iff \begin{cases} x'(t) = u^2(t) - v^2(t) \\ y'(t) = 2\,u(t)v(t) \\ \sigma(t) = u^2(t) + v^2(t) \end{cases}$$

for some polynomials $u(t)$, $v(t)$. The *complex–variable model* [7] for planar PH curves succinctly embodies this condition: identifying the point (x, y) with the complex number $x + \mathrm{i}\,y$, the Pythagorean hodograph structure is ensured by writing $\mathbf{r}'(t) = \mathbf{w}^2(t)$ for any complex polynomial $\mathbf{w}(t) = u(t) + \mathrm{i}\,v(t)$ with $\gcd(u(t), v(t)) = \mathrm{constant}$. The complex formulation greatly simplifies many basic algorithms [1, 8, 15, 18, 24, 30] for planar PH curves.

In the spatial case ($n = 3$), we need[2] *four* polynomials [4, 5] to characterize the Pythagorean nature of a hodograph $\mathbf{r}'(t) = (x'(t), y'(t), z'(t))$. Namely,

$$x'^2(t)+y'^2(t)+z'^2(t) = \sigma^2(t) \iff \begin{cases} x'(t) = u^2(t) + v^2(t) - p^2(t) - q^2(t) \\ y'(t) = 2\,[\,u(t)q(t) + v(t)p(t)\,] \\ z'(t) = 2\,[\,v(t)q(t) - u(t)p(t)\,] \\ \sigma(t) = u^2(t) + v^2(t) + p^2(t) + q^2(t) \end{cases}$$

for some polynomials $u(t)$, $v(t)$, $p(t)$, $q(t)$. The quaternion formulation for spatial Pythagorean hodographs, first introduced in [4], provides a very elegant and succinct embodiment of this structure. Quaternions can be represented as pairs of the form $\mathcal{A} = (a, \mathbf{a})$ and $\mathcal{B} = (b, \mathbf{b})$ where a, b are the *scalar parts* and $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$, $\mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$ are the *vector parts*. For brevity, we often simply write a for the "pure scalar" quaternion $(a, \mathbf{0})$ and $\mathbf{a}$ for the

[1] PH curves have also been defined in the *Minkowski metric* of relativity theory [4, 29]: such "MPH curves" play an important role in reconstructing the boundary of a shape from its medial axis transform.

[2] An earlier formulation [20] employing only three polynomials provides a sufficient, but not necessary, characterization of spatial Pythagorean hodographs (this characterization is not rotation–invariant).

"pure vector" quaternion $(0, \mathbf{a})$. The sum and product of $\mathcal{A}$, $\mathcal{B}$ are given by

$$\mathcal{A} + \mathcal{B} = (a + b, \mathbf{a} + \mathbf{b}) \,, \quad \mathcal{A}\,\mathcal{B} = (ab - \mathbf{a} \cdot \mathbf{b}, a\,\mathbf{b} + b\,\mathbf{a} + \mathbf{a} \times \mathbf{b}) \,.$$

Note that the product is non–commutative (i.e., $\mathcal{B}\mathcal{A} \neq \mathcal{A}\mathcal{B}$ in general).

Now if $\mathcal{A}(t) = u(t) + v(t)\mathbf{i} + p(t)\mathbf{j} + q(t)\mathbf{k}$ is a quaternion polynomial, and $\mathcal{A}^*(t) = u(t) - v(t)\mathbf{i} - p(t)\mathbf{j} - q(t)\mathbf{k}$ is its *conjugate*, the product

$$\begin{aligned} \mathbf{r}'(t) \;&=\; \mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}^*(t) \;=\; [\,u^2(t) + v^2(t) - p^2(t) - q^2(t)\,]\,\mathbf{i} \\ &+\; 2\,[\,u(t)q(t) + v(t)p(t)\,]\,\mathbf{j} \;+\; 2\,[\,v(t)q(t) - u(t)p(t)\,]\,\mathbf{k} \end{aligned} \qquad (1)$$

generates the PH structure in $\mathbb{R}^3$ ($\mathbf{j}$ or $\mathbf{k}$ can be interposed between $\mathcal{A}(t)$, $\mathcal{A}^*(t)$ in place of $\mathbf{i}$, yielding a permutation of $u(t)$, $v(t)$, $p(t)$, $q(t)$). We may express (1) as $\mathbf{r}'(t) = |\mathcal{A}(t)|^2\,\mathcal{U}(t)\,\mathbf{i}\,\mathcal{U}^*(t)$, where $|\mathcal{A}(t)|^2 = \mathcal{A}(t)\mathcal{A}^*(t)$ and $\mathcal{U}(t) = (\cos\frac{1}{2}\theta(t), \sin\frac{1}{2}\theta(t)\,\mathbf{n}(t))$ defines a *unit quaternion*, expressed in terms of an angle $\theta(t)$ and a unit vector $\mathbf{n}(t)$. The product $\mathcal{U}(t)\,\mathbf{i}\;\mathcal{U}^*(t)$ defines a *spatial rotation* of the basis vector $\mathbf{i}$ by angle $\theta(t)$ about the axis vector $\mathbf{n}(t)$, while the factor $|\mathcal{A}(t)|^2$ imposes a *scaling* of this rotated vector. Thus, we can interpret the form (1) as generating a spatial hodograph through a continuous family of spatial rotations and scalings of the basis vector $\mathbf{i}$.

An important feature of the form (1) is its *structural invariance* [11] under arbitrary spatial rotations of the coordinate system.[3] Namely, if the coordinate system $\tilde{\mathbf{r}} = (\tilde{x}, \tilde{y}, \tilde{z})$ is obtained from $\mathbf{r} = (x, y, z)$ by a rotation through angle ϕ about the unit vector $\mathbf{n} = n_x\mathbf{i} + n_y\mathbf{j} + n_z\mathbf{k}$, the hodograph in the new coordinate system becomes $\tilde{\mathbf{r}}'(t) = \tilde{\mathcal{A}}(t)\,\mathbf{i}\,\tilde{\mathcal{A}}(t)$, where $\tilde{\mathcal{A}}(t) = \mathcal{U}\,\mathcal{A}(t)$ with $\mathcal{U} = (\cos\frac{1}{2}\phi, \sin\frac{1}{2}\phi\,\mathbf{n})$. The components $\tilde{u}$, $\tilde{v}$, $\tilde{p}$, $\tilde{q}$ of $\tilde{\mathcal{A}}$ can be expressed in terms of those of $\mathcal{A}$ in matrix form as

$$\begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{p} \\ \tilde{q} \end{bmatrix} = \begin{bmatrix} \cos\frac{1}{2}\phi & -n_x\sin\frac{1}{2}\phi & -n_y\sin\frac{1}{2}\phi & -n_z\sin\frac{1}{2}\phi \\ n_x\sin\frac{1}{2}\phi & \cos\frac{1}{2}\phi & -n_z\sin\frac{1}{2}\phi & n_y\sin\frac{1}{2}\phi \\ n_y\sin\frac{1}{2}\phi & n_z\sin\frac{1}{2}\phi & \cos\frac{1}{2}\phi & -n_x\sin\frac{1}{2}\phi \\ n_z\sin\frac{1}{2}\phi & -n_y\sin\frac{1}{2}\phi & n_x\sin\frac{1}{2}\phi & \cos\frac{1}{2}\phi \end{bmatrix} \begin{bmatrix} u \\ v \\ p \\ q \end{bmatrix} .$$

Adoption of the quaternion model for spatial PH curves greatly facilitates the formulation and solution of key problems in their construction and analysis, and offers new theoretical insights. Compared to the complex–number model for planar PH curves, however, it requires greater care and attention to detail in its use, due to the non–commutative nature of the quaternion product.

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[3]This is essential for a characterization of spatial Pythagorean hodographs to be sufficient and necessary, and distinguishes (1) from an earlier formulation [20], which is sufficient only.

3. First-order spatial PH quintic Hermite interpolants

A key algorithm [12] in the construction of spatial PH curves is concerned with the problem of first–order Hermite interpolation — i.e., interpolation of given end points $\mathbf{p}_0$, $\mathbf{p}_1$ and derivatives $\mathbf{d}_0$, $\mathbf{d}_1$ by a spatial PH curve $\mathbf{r}(t)$ for $t \in [\,0, 1\,]$. The lowest–order PH curves capable of solving this problem for arbitrary spatial data are — as with planar PH curves — quintics. Whereas the planar PH quintic Hermite interpolation problem yields *four distinct solutions* [18], interpolation by the spatial PH quintics incurs a *two–parameter family of solutions* [12]. The shape of these interpolants may depend rather sensitively on these two free parameters, and the question of choosing "optimal" values for them is still an open problem (one possibility is to impose an additional constraint, such as a helicity condition [14], on the interpolants).

To construct spatial PH quintic Hermite interpolants, we begin by inserting a quadratic quaternion polynomial

$$\mathcal{A}(t) \,=\, \mathcal{A}_0(1-t)^2 \,+\, \mathcal{A}_1 2(1-t)t \,+\, \mathcal{A}_2 t^2$$

into the representation (1). Here the quaternion coefficients $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$ are to be determined by matching the Hermite data $\mathbf{p}_0$, $\mathbf{d}_0$ and $\mathbf{p}_1$, $\mathbf{d}_1$. The conditions $\mathbf{r}'(0) = \mathbf{d}_0$, $\mathbf{r}'(1) = \mathbf{d}_1$, and $\int_0^1 \mathbf{r}'(t)\,\mathrm{d}t = \mathbf{p}_1 - \mathbf{p}_0$ thus yield [12] the system of three equations

$$\mathcal{A}_0\,\mathbf{i}\,\mathcal{A}_0^* \,=\, \mathbf{d}_0\,, \qquad \mathcal{A}_2\,\mathbf{i}\,\mathcal{A}_2^* \,=\, \mathbf{d}_1\,, \tag{2}$$

$$\begin{aligned}(3\mathcal{A}_0 + 4\mathcal{A}_1 + 3\mathcal{A}_2)\;\mathbf{i}\;(3\mathcal{A}_0 + 4\mathcal{A}_1 + 3\mathcal{A}_2)^* \\ =\, 120(\mathbf{p}_1 - \mathbf{p}_0) \,-\, 15(\mathbf{d}_0 + \mathbf{d}_1) \,+\, 5(\mathcal{A}_0\,\mathbf{i}\,\mathcal{A}_2^* + \mathcal{A}_2\,\mathbf{i}\,\mathcal{A}_0^*)\end{aligned} \tag{3}$$

for $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{A}_2$. This system may be solved by noting that the equation

$$\mathcal{A}\,\mathbf{i}\,\mathcal{A}^* \,=\, \mathbf{d} \tag{4}$$

for a given vector $\mathbf{d} = |\mathbf{d}|(\lambda, \mu, \nu)$ admits a one–parameter family of solutions

$$\mathcal{A}(\phi) \,=\, \sqrt{\tfrac{1}{2}(1+\lambda)|\mathbf{d}|}\,\left(-\sin\phi\,,\; \frac{\cos\phi\,(\mathbf{i} + \mu\mathbf{j} + \nu\mathbf{k}) + \sin\phi\,(\nu\mathbf{j} - \mu\mathbf{k})}{1+\lambda}\right)$$

where ϕ is a free angular variable. The quaternion $\mathcal{A}$ serves to scale/rotate the basis vector $\mathbf{i}$ into the given vector $\mathbf{d}$ — the appearance of a free parameter in the solution reflects the fact that, in $\mathbb{R}^3$, there is a *continuous family of spatial rotations* that will map one unit vector into another.

Equations (2) can be solved directly for $\mathcal{A}_0, \mathcal{A}_2$ using the known form of the solution to (4). These quaternions depend on free parameters, ϕ_0 and ϕ_2 say. Substituting them into (2) we may determine $3\mathcal{A}_0 + 4\mathcal{A}_1 + 3\mathcal{A}_2$, and hence $\mathcal{A}_1$, using the solution to (4). Again, this incurs a new free parameter — ϕ_1, say. Although the complete solution incurs three indeterminate angular variables ϕ_0, ϕ_1, ϕ_2, close inspection reveals [12] that the Hermite interpolants depend only upon the *differences* of these angles. Hence, we may take $\phi_1 = 0$ without loss of generality, and the Hermite interpolants to $\mathbf{p}_0, \mathbf{p}_1$ and $\mathbf{d}_0, \mathbf{d}_1$ comprise a two–parameter family. Once $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$ are known, the Bezier control points of the interpolant are given by

$$\begin{aligned}
\mathbf{p}_1 &= \mathbf{p}_0 + \frac{1}{5}\mathcal{A}_0\,\mathbf{i}\,\mathcal{A}_0^*\,,\\
\mathbf{p}_2 &= \mathbf{p}_1 + \frac{1}{10}(\mathcal{A}_0\,\mathbf{i}\,\mathcal{A}_1^* + \mathcal{A}_1\,\mathbf{i}\,\mathcal{A}_0^*)\,,\\
\mathbf{p}_3 &= \mathbf{p}_2 + \frac{1}{30}(\mathcal{A}_0\,\mathbf{i}\,\mathcal{A}_2^* + 4\,\mathcal{A}_1\,\mathbf{i}\,\mathcal{A}_1^* + \mathcal{A}_2\,\mathbf{i}\,\mathcal{A}_0^*)\,,\\
\mathbf{p}_4 &= \mathbf{p}_3 + \frac{1}{10}(\mathcal{A}_1\,\mathbf{i}\,\mathcal{A}_2^* + \mathcal{A}_2\,\mathbf{i}\,\mathcal{A}_1^*)\,,\\
\mathbf{p}_5 &= \mathbf{p}_4 + \frac{1}{5}\mathcal{A}_2\,\mathbf{i}\,\mathcal{A}_2^*\,,
\end{aligned}$$

with $\mathbf{p}_0$ being an arbitrary integration constant. Examples of spatial PH quintic Hermite interpolants, for specific ϕ_0, ϕ_2 values, are shown in Figure 1.

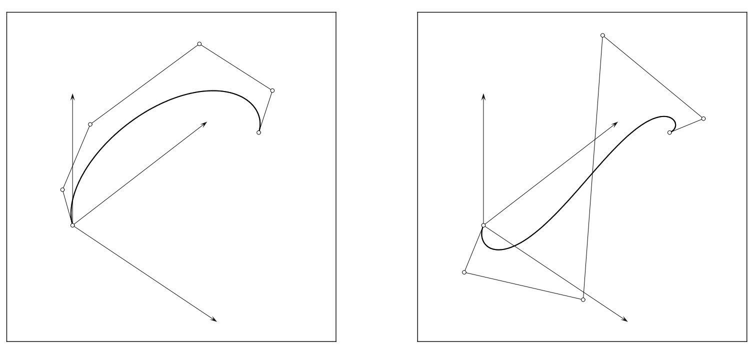

Figure 1. Examples of spatial PH quintics constructed as first–order Hermite interpolants.

A challenging open problem is to generalize the formulation and solution of the two–point Hermite interpolation to the smooth interpolation of $N+1$ points $\mathbf{p}_0, \ldots, \mathbf{p}_N$ in $\mathbb{R}^3$ by C^2 spatial PH quintic splines. In the planar case, the PH spline problem incurs solution of a "tridiagonal" system of N quadratic equations in N complex unknowns. An analogous quaternion system can be formulated in the spatial case [17]. In attempting to solve it, however, one must take full account of the non–commutative nature of the quaternion product, and the residual freedoms associated with each spline segment.

4. Rotation–minimizing frames on spatial PH curves

An *adapted frame* along a space curve $\mathbf{r}(t)$ is a right–handed system of three mutually orthogonal unit vectors $(\mathbf{t}, \mathbf{e}_1, \mathbf{e}_2)$ of which $\mathbf{t} = \mathbf{r}'/|\mathbf{r}'|$ is the tangent vector, and $\mathbf{e}_1$, $\mathbf{e}_2$ span the normal plane at each point such that $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{t}$. The most familiar example is the *Frenet frame* $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ comprising the tangent, normal $\mathbf{n}$ (pointing to the center of curvature), and binormal $\mathbf{b} = \mathbf{t} \times \mathbf{n}$. The variation of the Frenet frame with arc length is described [28] by the equations

$$\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} = \mathbf{d} \times \mathbf{t}\,, \qquad \frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s} = \mathbf{d} \times \mathbf{n}\,, \qquad \frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s} = \mathbf{d} \times \mathbf{b}\,, \tag{5}$$

where the *Darboux vector* is given in terms of the curvature κ and torsion τ by

$$\mathbf{d} = \kappa\,\mathbf{b} + \tau\,\mathbf{t}\,. \tag{6}$$

Equations (5) characterize the instantaneous variation of the Frenet frame as a rotation about the vector $\mathbf{d}$, at a rate given by the "total curvature"

$$\omega = |\mathbf{d}| = \sqrt{\kappa^2 + \tau^2}\,.$$

However, the Frenet frame is often unsuitable for use as an adapted frame in applications such as geometric design, computer graphics, animation, motion planning, and robotics. The vectors $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ do not, in general, have a rational dependence on the curve parameter t, and at inflection points (where $\kappa = 0$) $\mathbf{n}$ and $\mathbf{b}$ may suffer sudden inversions. Furthermore, the component $\tau\,\mathbf{t}$ of the instantaneous rotation vector (6) corresponds to an "unnecessary" rotation in the curve normal plane, that yields undesirable results in computer animation, swept surface constructions, and motion planning. Among the many adapted frames [2] on a space curve, Klok [27] suggests the *rotation–minimizing frame* (RMF) as the most suitable for such applications. The RMF is defined so as to "cancel" the $\tau\,\mathbf{t}$ component of the rotation vector by setting

$$\begin{bmatrix} \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{n} \\ \mathbf{b} \end{bmatrix},$$

where the angular function $\theta(t)$ is defined[4] [22] by

$$\theta(t) = \theta_0 - \int_0^t \tau(u)\,|\mathbf{r}'(u)|\,\mathrm{d}u\,. \tag{7}$$

Because this integral does not admit a closed–form reduction for "ordinary" polynomial and rational curves, schemes have been proposed to approximate

[4] An incorrect sign before the integral is given in [22].

RMFs or to approximate given curves by “simple” segments (e.g., cicular arcs) with known RMFs [23, 25, 26, 33].

For PH curves, the integrand in (7) is a *rational function* and thus admits closed–form integration [9]. A simplification of this integral arises through the fact that PH curves exhibit the remarkable factorization

$$|\mathbf{r}' \times \mathbf{r}''|^2 \;=\; \sigma^2 \rho \,,$$

where $\sigma = u^2 + v^2 + p^2 + q^2$, and ρ is the polynomial defined by

$$\begin{aligned}\rho \;=\; 4\,[\,(up' - u'p)^2 \;+\; (uq' - u'q)^2 \;+\; (vp' - v'p)^2 \;+\; (vq' - v'q)^2 \\ +\, 2(uv' - u'v)(pq' - p'q)\,]\,.\end{aligned}$$

Thus, for a PH curve, we have

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} \;=\; -\,\frac{[\,\mathbf{r}'(t) \times \mathbf{r}''(t)\,] \cdot \mathbf{r}'''(t)}{\sigma(t)\,\rho(t)}\,.$$

For PH quintics, $(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''$ is of degree 6, while σ and ρ are both quartic in t. The latter must be factorized to perform a partial fraction decomposition of the integrand: this can be accomplished by Ferrari’s method [32]. Complete details on the closed–form integration of this equation may be found in [9].

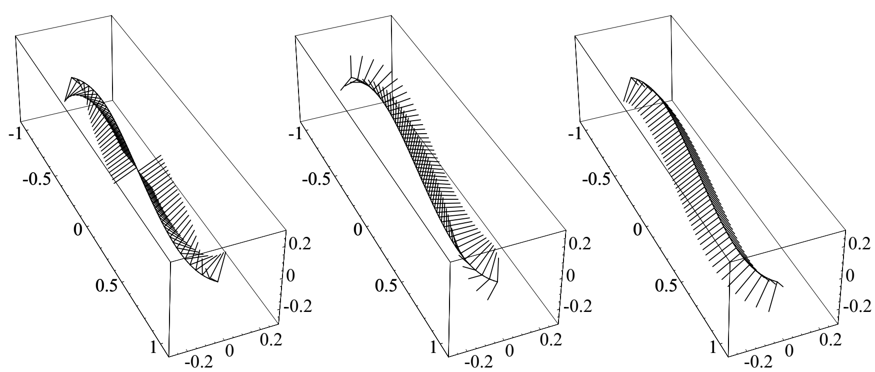

Figure 2. Comparison of Frenet frame (left), Euler–Rodrigues frame (center), and the rational approximation of rotation–minimizing frame (right) on a spatial PH quintic (for clarity, the tangent vector is omitted). Note the sudden reversal of the Frenet frame at the inflection point.

Since the integral in (7) involves rational and logarithmic terms, we describe in [13] an alternative rational approximation scheme, based on the equation

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} \;=\; 2\,\frac{u'v - uv' - p'q + pq'}{u^2 + v^2 + p^2 + q^2}$$

characterizing the variation of the RMF relative to the *Euler–Rodrigues frame* (ERF), defined [3] by

$$\mathbf{t}(t) = \frac{\mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}^*(t)}{\mathcal{A}(t)\mathcal{A}^*(t)}\,, \quad \mathbf{u}(t) = \frac{\mathcal{A}(t)\,\mathbf{j}\,\mathcal{A}^*(t)}{\mathcal{A}(t)\mathcal{A}^*(t)}\,, \quad \mathbf{v}(t) = \frac{\mathcal{A}(t)\,\mathbf{k}\,\mathcal{A}^*(t)}{\mathcal{A}(t)\mathcal{A}^*(t)}\,.$$

The ERF is a rational adapted frame defined on spatial PH curves. In Figure 2 we compare the Frenet frame, ERF, and rational RMF approximation on an inflectional PH quintic — the superior behavior of the RMF is clearly apparent.

A natural question is when (or whether) one can have a *rational* RMF on a given non–planar polynomial curve. Note that the curve must be a PH curve to have a rational adapted frame, since only PH curves have rational unit tangents. A partial answer [3] to this question can be given in terms of the ERF, defined above — the minimum degree of non–planar PH curves that have rotation–minimizing ERFs is seven.

5. Tangent indicatrix uniquely determines PH curves

The hodograph $\mathbf{r}'(t)$ of a parametric curve can be expressed as the product of a scalar magnitude and a unit vector

$$\mathbf{r}'(t) \;=\; \sigma(t)\,\mathbf{t}(t)\,,$$

both dependent on the curve parameter t. As noted above, $\sigma(t) = |\mathbf{r}'(t)|$ is the parametric speed (the derivative of arc length s with respect to t). The vector $\mathbf{t}(t) = \mathbf{r}'(t)/\sigma(t)$ traces a locus on the unit sphere, the *tangent indicatrix* of the curve. Whereas the parametric speed specifies the *magnitude* of the hodograph $\mathbf{r}'(t)$ at each point, the tangent indicatrix indicates its *direction*. Integration of a hodograph yields a unique curve, modulo a translation corresponding to the integration constant. We will show that, for PH curves, $\sigma(t)$ plays a somewhat redundant role in the determination of a curve from its hodograph $\mathbf{r}'(t)$ — $\mathbf{r}(t)$ *is uniquely detemined* (modulo uniform scaling) *by the tangent indicatrix only*.

This property distinguishes PH curves from "ordinary" polynomials curves, for which both the parametric speed and tangent indicatrix influence the shape of the curve $\mathbf{r}(t)$ obtained by integration of the hodograph $\mathbf{r}'(t) = \sigma(t)\,\mathbf{t}(t)$. Two ordinary polynomial curves with the same tangent indicatrix but different parametric speeds have, in general, quite different shapes. Note also that, for PH curves, the tangent indicatrix $\mathbf{t}(t) = \mathbf{r}'(t)/\sigma(t)$ is a *rational* curve on the unit sphere, since $\sigma(t)$ is a polynomial (whereas, for an ordinary polynomial curve, it is the square root of a polynomial, and hence $\mathbf{t}(t)$ is not rational).

Proposition 1. *Let $\mathbf{r}(t)$, $\tilde{\mathbf{r}}(t)$ be two polynomial PH curves whose hodographs have relatively prime components. If these curves possess the same tangent indicatrix, they differ by at most a translation and uniform scaling — i.e., $\mathbf{r}'(t) = \gamma\,\tilde{\mathbf{r}}'(t)$ for some $\gamma \neq 0$.*

Proof. Since $\mathbf{r}(t) = (x(t), y(t), z(t))$ and $\tilde{\mathbf{r}}(t) = (\tilde{x}(t), \tilde{y}(t), \tilde{z}(t))$ are both PH curves, polynomials $\sigma(t)$ and $\tilde{\sigma}(t)$ exist such that their hodographs

$$\mathbf{r}'(t) = (x'(t), y'(t), z'(t)) \quad \text{and} \quad \tilde{\mathbf{r}}'(t) = (\tilde{x}'(t), \tilde{y}'(t), \tilde{z}'(t))$$

satisfy

$$x'^2(t) + y'^2(t) + z'^2(t) = \sigma^2(t) \quad \text{and} \quad \tilde{x}'^2(t) + \tilde{y}'^2(t) + \tilde{z}'^2(t) = \tilde{\sigma}^2(t)\,.$$

Furthermore, since $x'(t), y'(t), z'(t)$ and $\tilde{x}'(t), \tilde{y}'(t), \tilde{z}'(t)$ are relatively prime, $\sigma(t)$ and $\tilde{\sigma}(t)$ never vanish, and we can assume $\sigma(t) > 0$ and $\tilde{\sigma}(t) > 0$ for all t. The tangent indicatrices of $\mathbf{r}(t)$ and $\tilde{\mathbf{r}}(t)$ are then $\mathbf{t}(t) = \mathbf{r}'(t)/\sigma(t)$ and $\tilde{\mathbf{t}}(t) = \tilde{\mathbf{r}}'(t)/\tilde{\sigma}(t)$, and if they are the same we must have

$$x'(t) = \frac{\sigma(t)\tilde{x}'(t)}{\tilde{\sigma}(t)}\,, \quad y'(t) = \frac{\sigma(t)\tilde{y}'(t)}{\tilde{\sigma}(t)}\,, \quad z'(t) = \frac{\sigma(t)\tilde{z}'(t)}{\tilde{\sigma}(t)}\,.$$

We claim that $\tilde{\sigma}(t)$ divides into $\sigma(t)$. The Proposition then follows, since we can swap the roles of $\sigma(t)$ and $\tilde{\sigma}(t)$, and thus $\sigma(t) = \gamma\,\tilde{\sigma}(t)$ for some $\gamma \neq 0$.

Consider the equation $x'(t) = \sigma(t)\tilde{x}'(t)/\tilde{\sigma}(t)$. Since $x'(t)$ is a polynomial, each non–constant factor of $\tilde{\sigma}(t)$ must divide into either $\sigma(t)$ or $\tilde{x}'(t)$, and an analogous statement holds for $y'(t)$ and $z'(t)$. However, if we postulate that a non–constant factor of $\tilde{\sigma}(t)$ divides into $\tilde{x}'(t)$, but not $\sigma(t)$, we must conclude that this factor also divides into $\tilde{y}'(t)$ and $\tilde{z}'(t)$, which contradicts the fact that $\gcd(\tilde{x}'(t), \tilde{y}'(t), \tilde{z}'(t)) = \text{constant}$. Hence, we may deduce that $\sigma(t) = \gamma\,\tilde{\sigma}(t)$ for some $\gamma \neq 0$. ☐

Corollary 1. *Given a rational curve* $\mathbf{q}(t) = (a(t), b(t), c(t))/\sigma(t)$ *on the unit sphere in* $\mathbb{R}^3$ *satisfying* $\gcd(a(t), b(t), c(t)) = \text{constant}$*, there is — modulo uniform scaling and translation — a unique polynomial PH curve* $\mathbf{r}(t) = (x(t), y(t), z(t))$ *with* $\gcd(x'(t), y'(t), z'(t)) = \text{constant}$ *that has* $\mathbf{q}(t)$ *as its tangent indicatrix.*

Proof. Since $\mathbf{q}(t)$ lies on the unit sphere, we have $a^2(t)+b^2(t)+c^2(t) = \sigma^2(t)$. Since $\gcd(a(t), b(t), c(t)) = \text{constant}$, $\sigma(t)$ is never zero, and thus we may assume $\sigma(t) > 0$ for all t. Clearly, for any scalar factor $\gamma \neq 0$ and integration constant $\mathbf{r}_0$,

$$\mathbf{r}(t) = \gamma \int \sigma(t)\mathbf{q}(t)\;\mathrm{d}t + \mathbf{r}_0$$

defines a PH curve, and the uniqueness of $\mathbf{r}(t)$, modulo the uniform scaling γ and translation $\mathbf{r}_0$, follows from Proposition 1. ☐

Although we have phrased the above results in terms of spatial PH curves, they obviously also apply to planar PH curves (for which the tangent indicatrix lies on the unit circle, rather than the unit sphere).

6. Inversion of spatial Pythagorean hodographs

In [7] we addressed the question of "how many" planar PH curves exist by showing that, in the plane, the infinite sets of regular PH curves and regular "ordinary" polynomial curves have the same cardinality — we can establish a *one–to–one correspondence* between their members (corresponding curves are of different degree). This was accomplished by invoking the map $\mathbf{z} \to \mathbf{z}^2$ and its inverse in the complex–variable model for planar PH curves.

In seeking an analogous result for spatial PH curves, we need the ability to invert the map $\mathcal{A}(t) \to \mathbf{r}'(t)$ defined by (1) — i.e., given a spatial Pythagorean hodograph $\mathbf{r}'(t)$, we wish to identify the pre–image curve(s) $\mathcal{A}(t)$ in quaternion space that generate it through expression (1). In §3 we gave the general solution to the analogous equation (4) for a fixed vector $\mathbf{d}$. However, this solution is not appropriate to the problem of inverting (1), since if we replace $\mathbf{d}$ by $\mathbf{r}'(t)$ it exhibits the factor $\sqrt{|\mathbf{r}'(t)|}$ and we require a *polynomial* pre–image $\mathcal{A}(t)$.

Dietz et al. [5] give a constructive proof for the existence of polynomials $u(t), v(t), p(t), q(t)$ satisfying (1) for a given $\mathbf{r}'(t)$ by invoking the *generalized stereographic projection*. Since it is not convenient for actually computing the pre–image, our goal here is to re–work this proof into a practical algorithm.

Generalized stereographic projection

The notation in [5] differs somewhat from our quaternion model. In [5] (x_0, x_1, x_2, x_3) denote homogeneous coordinates in 3–dimensional projective real space $\mathbb{RP}^3$, and the hodograph $\mathbf{r}'(t) = (x'(t), y'(t), z'(t))$ is mapped to its tangent indicatrix by the correspondence $x_0 = \sigma$, $x_1 = x'$, $x_2 = y'$, $x_3 = z'$.

The generalized stereographic projection δ maps points $(p_0, p_1, p_2, p_3) \in \mathbb{RP}^3$ to points (x_0, x_1, x_2, x_3) on the unit sphere (satisfying $x_1^2 + x_2^2 + x_3^2 = x_0^2$) according to

$$\begin{aligned}
x_0 &= p_0^2 + p_1^2 + p_2^2 + p_3^2 \,, \\
x_1 &= 2p_0p_1 - 2p_2p_3 \,, \\
x_2 &= 2p_1p_3 + 2p_0p_2 \,, \\
x_3 &= p_1^2 + p_2^2 - p_0^2 - p_3^2 \,.
\end{aligned}$$

This is equivalent to the quaternion formulation if we identify

$$(x', y', z', \sigma) = (x_3, x_2, x_1, x_0) \quad \text{and} \quad (u, v, p, q) = (p_2, p_1, p_3, p_0) \,.$$

Now suppose we are given a Pythagorean hodograph $\mathbf{r}'(t)$ with relatively prime coordinate components in $\mathbb{R}[t]$. Then the tetrad members x_0, x_1, x_2, x_3 are also relatively prime in $\mathbb{R}[t]$. We describe how to compute a pre–image p_0, p_1, p_2, p_3 under the above quadratic transformation.

1 Re–write the condition $x_1^2 + x_2^2 + x_3^2 = x_0^2$ as $(x_1 + \mathrm{i}x_2)(x_1 - \mathrm{i}x_2) = (x_0 + x_3)(x_0 - x_3)$.

2 Compute $d = \gcd(x_1, x_2)$ in $\mathbb{R}[t]$ such that $x_1 + \mathrm{i}x_2 = d(\tilde{x} + \mathrm{i}\tilde{x}_2)$ and $\gcd(\tilde{x}_1, \tilde{x}_2) = 1$ in $\mathbb{R}[t]$.

3 Factorize d in $\mathbb{R}[t]$ and $\tilde{x}_1 + \mathrm{i}\tilde{x}_2$ in $\mathbb{C}[t]$.

4 Let r be a factor of d in $\mathbb{R}[t]$. We claim that r divide into either $x_0 + x_3$ or $x_0 - x_3$, but not both. For, if r divides into both it must be a real common factor of x_0 and x_3, but r is already a real common factor of x_1 and x_2, which contradicts the fact that x_0, x_1, x_2, x_3 are relatively prime in $\mathbb{R}[t]$. Hence, we can write $d = r_1 \cdots r_m s_1 \cdots s_n$ such that the r_i's are factors of $x_0 + x_3$ and the s_i's are factors of $x_0 - x_3$.

5 Let $\gamma_1 \ldots \gamma_\kappa$ be a prime factor decomposition of $\tilde{x}_1 + \mathrm{i}\tilde{x}_2$ in $\mathbb{C}[t]$ (note that some factors may be repeated). Since we have already removed all the real factors, there is no pair $i \neq j$ such that $\gamma_i = \overline{\gamma}_j$. Since $\mathbb{C}[t]$ is a unique factorization domain, we can uniquely arrange the decomposition into groups $\alpha_1 \ldots \alpha_\rho$ and $\beta_1 \ldots \beta_\sigma$ such that the first divides into $x_0 + x_3$ and the second $x_0 - x_3$. Note that if γ_i divides into $x_0 + x_3$, so does $\overline{\gamma}_i$.

6 From the preceding two steps, we have $x_0 + x_3 = (r_1 \cdots r_m)^2 |\alpha_1 \cdots \alpha_\rho|^2$ and $x_0 - x_3 = (s_1 \cdots s_n)^2 |\beta_1 \cdots \beta_\sigma|^2$. Defining $\phi = r_1 \cdots r_m \alpha_1 \cdots \alpha_\rho$ and $\psi = s_1 \cdots s_n \beta_1 \cdots \beta_\sigma$, one can verify that $p_0 = \mathrm{Re}(\psi)$, $p_1 = \mathrm{Re}(\phi)$, $p_2 = \mathrm{Im}(\phi)$, $p_3 = \mathrm{Im}(\psi)$ constitute a solution (in fact $p_i/\sqrt{2}$ gives the exact form but we ignored the common constant multiple).

For example, consider the case

$$\begin{aligned} x_0 &= 7 - 8\,t + 13\,t^2 + 2\,t^3 + 4\,t^4\,, \\ x_1 &= -6 + 8\,t - 4\,t^2 + 10\,t^3\,, \\ x_2 &= 2 + 8\,t + 2\,t^3 + 4\,t^4\,, \\ x_3 &= 3 - 8\,t + t^2 + 2\,t^3\,. \end{aligned}$$

Then x_1 and x_2 are relatively prime in $\mathbb{R}[t]$, and we have

$$x_1 + \mathrm{i}x_2 \;=\; (t + 1 - 2\mathrm{i})(2t - 1 + \mathrm{i})(t - \mathrm{i}\tfrac{1}{2}(1 - \sqrt{5}))(2\mathrm{i}\,t + 1 + \sqrt{5})\,,$$

$$x_0 + x_3 \;=\; (t + 1 + 2\mathrm{i})(t + 1 - 2\mathrm{i})(2t - 1 + \mathrm{i})(2t - 1 - \mathrm{i})\,.$$

Hence, we can set $\alpha_1 = t + 1 - 2\mathrm{i}$, $\alpha_2 = 2t - 1 + \mathrm{i}$, $\beta_1 = 2\mathrm{i}\,t + 1 + \sqrt{5}$, $\beta_2 = t - \mathrm{i}(1 - \sqrt{5})/2$. Then $\phi = 1 + t + 2t^2 + \mathrm{i}3(1 - t)$ and $\psi = 2t + \mathrm{i}2(1 + t^2)$, giving $p_0 = \sqrt{2}t$, $p_1 = (1 + t + 2t^2)/\sqrt{2}$, $p_2 = 3(1 - t)/\sqrt{2}$, $p_3 = \sqrt{2}(1 + t^2)$.

However, this solution is not unique. In fact x_0, x_1, x_2, x_3 were initially constructed using $p_0 = 1 + t + t^2$, $p_1 = -1 + 2t + t^2$, $p_2 = 2 - t + t^2$, $p_3 = 1 - t + t^2$. The non–uniqueness has also been noted in the quaternion formulation [11] — if we define

$$\begin{bmatrix} \hat{u}(t) \\ \hat{v}(t) \\ \hat{p}(t) \\ \hat{q}(t) \end{bmatrix} = \begin{bmatrix} \cos\xi & -\sin\xi & 0 & 0 \\ \sin\xi & \cos\xi & 0 & 0 \\ 0 & 0 & \cos\xi & \sin\xi \\ 0 & 0 & -\sin\xi & \cos\xi \end{bmatrix} \begin{bmatrix} u(t) \\ v(t) \\ p(t) \\ q(t) \end{bmatrix}$$

then $\hat{u}(t)$, $\hat{v}(t)$, $\hat{p}(t)$, $\hat{q}(t)$ and $u(t)$, $v(t)$, $p(t)$, $q(t)$ define the same PH curve.

This is also apparent from the generalized stereographic projection. The pre–image $\delta^{-1}(Q)$ of any point $Q = (x_0, x_1, x_2, x_3)$ on the sphere comprises the straight line $\lambda A + \mu B$ in $\mathbb{RP}^3$, where $A = (x_2, 0, x_0 + x_3, -x_1)$, $B = (x_1, x_0 + x_3, 0, x_2)$ and $\lambda, \mu \in \mathbb{R}$. In fact, one can verify that

$$\delta(\lambda A + \mu B) \;=\; 2(\lambda^2 + \mu^2)(x_0 + x_3)(x_0, x_1, x_2, x_3)\,.$$

Now suppose we have a degree–$2n$ curve $x(t) = (x_0(t), x_1(t), x_2(t), x_3(t))$ on the unit sphere such that $x_0(t), \ldots, x_3(t)$ are relatively prime in $\mathbb{R}[t]$. The pre–image of this curve under the generalized stereographic projection, i.e., the set $S = \{\delta^{-1}(x(t)) : t \in \mathbb{R}\}$, is clearly a ruled surface with base curves given by $A(t) = (x_2(t), 0, x_0(t) + x_3(t), -x_1(t))$ and $B(t) = (x_1(t), x_0(t) + x_3(t), 0, x_2(t))$. Note that these two base curves are also of degree $2n$.

However, a theorem of Dietz et al. [5] states that we can always find another pair $\{P(t), \tilde{P}(t)\}$ of base curves whose degree is just n. If we take $P(t) = (p_0(t), p_1(t), p_2(t), p_3(t))$, we can take $\tilde{P}(t) = (-p_3(t), p_2(t), -p_1(t), p_0(t))$. For each t, the four points $A(t)$, $B(t)$, $P(t)$, $\tilde{P}(t)$ are collinear.

7. Closure

The quaternion formulation for spatial PH curves, first introduced by Choi et al. [4], has paved the way for development of basic algorithms concerned with their construction, analysis, and applications. In this paper, we surveyed these new developments and identified a number of important open problems. Compared to planar PH curves, the construction of spatial PH curves typically incurs free parameters. A deeper theoretical understanding of the role of these parameters, and their optimal selection, remains to be achieved.

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References

[1] G. Albrecht and R. T. Farouki (1996), Construction of C^2 Pythagorean hodograph interpolating splines by the homotopy method, *Adv. Comp. Math.* **5**, 417–442.

[2] R. L. Bishop (1975), There is more than one way to frame a curve, *Amer. Math. Monthly* **82**, 246–251.

[3] H. I. Choi and C. Y. Han (2002), Euler–Rodrigues frames on spatial Pythagorean–hodograph curves, *Comput. Aided Geom. Design* **19**, 603–620.

[4] H. I. Choi, D. S. Lee, and H. P. Moon (2002), Clifford algebra, spin representation, and rational parameterization of curves and surfaces, *Adv. Comp. Math.* **17**, 5–48.

[5] R. Dietz, J. Hoschek, and B. Jüttler (1993), An algebraic approach to curves and surfaces on the sphere and on other quadrics, *Comput. Aided Geom. Design* **10**, 211–229.

[6] R. T. Farouki (1992), Pythagorean–hodograph curves in practical use, *Geometry Processing for Design and Manufacturing* (R. E. Barnhill, ed.), SIAM, 3–33.

[7] R. T. Farouki (1994), The conformal map $z \to z^2$ of the hodograph plane, *Comput. Aided Geom. Design* **11**, 363–390.

[8] R. T. Farouki (1996), The elastic bending energy of Pythagorean hodograph curves, *Comput. Aided Geom. Design* **13**, 227–241.

[9] R. T. Farouki (2002), Exact rotation–minimizing frames for spatial Pythagorean–hodograph curves, *Graph. Models* **64**, 382–395.

[10] R. T. Farouki (2002), Pythagorean–hodograph curves, in *Handbook of Computer Aided Geometric Design* (G. Farin. J. Hoschek, and M–S. Kim, eds.), North Holland, 405–427.

[11] R. T. Farouki, M. al–Kandari, and T. Sakkalis (2002), Structural invariance of spatial Pythagorean hodographs, *Comput. Aided Geom. Design* **19**, 395–407.

[12] R. T. Farouki, M. al–Kandari, and T. Sakkalis (2002), Hermite interpolation by rotation–invariant spatial Pythagorean–hodograph curves, *Adv. Comp. Math.* **17**, 369–383.

[13] R. T. Farouki and C. Y. Han (2003), Rational approximation schemes for rotation–minimizing frames on Pythagorean–hodograph curves, *Comput. Aided Geom. Design* **20**, 435–454.

[14] R. T. Farouki, C. Y. Han, C. Manni, and A. Sestini (2003), Characterization and construction of helical Pythagorean-hodograph quintic space curves, *Journal of Computational and Applied Mathematics***162**, 365–392 (2004)

[15] R. T. Farouki, B. K. Kuspa, C. Manni, and A. Sestini (2001), Efficient solution of the complex quadratic tridiagonal system for C^2 PH quintic splines, *Numer. Algor.* **27**, 35–60.

[16] R. T. Farouki, J. Manjunathaiah, D. Nicholas, G.–F. Yuan, and S. Jee (1998), Variable feedrate CNC interpolators for constant material removal rates along Pythagorean–hodograph curves, *Comput. Aided Design* **30**, 631–640.

[17] R. T. Farouki, C. Manni, and A. Sestini (2003), Spatial C^2 PH quintic splines, *Curve and Surface Design: Saint Malo 2002* (T. Lyche, M.-L. Mazure, and L. L. Schumaker, eds.), Nashboro Press, pp. 147–156.

[18] R. T. Farouki and C. A. Neff (1995), Hermite interpolation by Pythagorean hodograph quintics, *Math. Comp.* **64**, 1589–1609.

[19] R. T. Farouki and T. Sakkalis (1990), Pythagorean hodographs, *IBM J. Res. Develop.* **34**, 736–752.

[20] R. T. Farouki and T. Sakkalis (1994), Pythagorean–hodograph space curves, *Adv. Comp. Math.* **2**, 41–66.

[21] R. T. Farouki and S. Shah (1996), Real–time CNC interpolators for Pythagorean–hodograph curves, *Comput. Aided Geom. Design* **13**, 583–600.

[22] H. Guggenheimer (1989), Computing frames along a trajectory, *Comput. Aided Geom. Design* **6**, 77–78.

[23] B. Jüttler (1998), Generating rational frames of space curves via Hermite interpolation with Pythagorean hodograph cubic splines, in *Geometric Modeling and Processing '98*, Bookplus Press, pp. 83–106.

[24] B. Jüttler (2001), Hermite interpolation by Pythagorean hodograph curves of degree seven, *Math. Comp.* **70**, 1089–1111.

[25] B. Jüttler and C. Maurer (1999), Cubic Pythagorean hodograph spline curves and applications to sweep surface modelling, *Comput. Aided Design* **31**, 73–83.

[26] B. Jüttler and C. Maurer (1999), Rational approximation of rotation minimizing frames using Pythagorean–hodograph cubics, *J. Geom. Graphics* **3**, 141–159.

[27] F. Klok (1986), Two moving coordinate frames for sweeping along a 3D trajectory, *Comput. Aided Geom. Design* **3**, 217–229.

[28] E. Kreyszig (1959), *Differential Geometry*, University of Toronto Press.

[29] H. P. Moon (1999), Minkowski Pythagorean hodographs, *Comput. Aided Geom. Design* **16**, 739–753.

[30] H. P. Moon, R. T. Farouki, and H. I. Choi (2001), Construction and shape analysis of PH quintic Hermite interpolants, *Comput. Aided Geom. Design* **18**, 93–115.

[31] Y–F. Tsai, R. T. Farouki, and B. Feldman (2001), Performance analysis of CNC interpolators for time–dependent feedrates along PH curves, *Comput. Aided Geom. Design* **18**, 245–265.

[32] J. V. Uspensky (1948), *Theory of Equations*, McGraw–Hill, New York.

[33] W. Wang and B. Joe (1997), Robust computation of the rotation minimizing frame for sweep surface modelling, *Comput. Aided Design* **29**, 379–391.