Propositions, \( P, Q, R, \ldots \) True or False

Predicates take \( n \) tuples as the subject

\( P(x), P(x, y), Q(x, y), \ldots \)

where \( x, y, z, \ldots \) are elements in a domain of discourse.

Existential quantifiers bind the predicates and make them into propositions

e.g. \( \exists x, \forall x, \exists x > 0, \forall x, \ldots \)

Implication \( p \rightarrow q \)

Antecedent/hypothesis

Consequent/conclusion

\( p \rightarrow q \) is only false when \( p = T \) and \( q = F \).

(Why contrapositive \( \neg q \rightarrow \neg p \) is logically equivalent.)

Typically interested in implications, why compound hypothesis are conclusions

E.g. \( (s \lor t) \land (r \lor u) \rightarrow q \)

\( P(x, y) \land Q(x, y) \rightarrow Z(x, y) \)

How to specify which elements \( x \) in \( U \) satisfy \( P(x) \)?

Answer: Today's lecture: Sets & Set Builder Notation
But first let's prove a logical equiv.

Recall De Morgan's laws

1) \( \neg(p \land q) \equiv \neg p \lor \neg q \)
2) \( \neg(p \lor q) \equiv \neg p \land \neg q \)

we can generalize to many variables

1) \( \neg(p_1 \land p_2 \land \ldots \land p_n) \equiv \neg p_1 \lor \neg p_2 \lor \ldots \lor \neg p_n \)
2) \( \neg(p_1 \lor p_2 \lor \ldots \lor p_n) \equiv \neg p_1 \land \neg p_2 \land \ldots \land \neg p_n \)

Let's use this to prove two relations from Lec 3.

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Assumed:

\( \forall x P(x) \equiv \neg \exists x \neg P(x) \) – prove this now

\( \exists x P(x) \equiv \neg \forall x \neg P(x) \) – on HW 2.

\( \forall x P(x) \equiv \neg \exists x \neg P(x) \)

Recall

\( \exists x P(x) \equiv P(x_1) \lor P(x_2) \lor \ldots \lor P(x_n) \)

\( \forall x P(x) \equiv P(x_1) \land P(x_2) \land \ldots \land P(x_n) \)

\( \neg \exists x \neg P(x) \equiv \neg (\neg P(x_1) \lor \neg P(x_2) \lor \ldots \lor \neg P(x_n)) \) – precedence of not operator

\( \equiv \neg (\neg P(x_1) \lor \neg P(x_2) \lor \ldots \lor \neg P(x_n)) \) – defn \( \equiv \)

\( \equiv P(x_1) \land P(x_2) \land \ldots \land P(x_n) \)

Demorgan's 2

\( \equiv \forall x P(x) \)

\( \equiv \forall x P(x) \)
Specifying Sets

- Domain/universe of discourse.
  (e.g. people, integers, real #s, etc.)
  Let us denote elements of \( U \)
  \( x_1, x_2, x_3, \ldots, x_n \)

  \( S \) is a subset of the \( x_i \)'s that have some property in common.

  Typically \( S \) as the set of \( x_i \)'s for which \( P(x) = T \).

Two ways to specify sets

- explicit enumeration
  \( S = \{ x_1, x_2 \} \) or \( S = \{ x_5, x_7, x_9, x_{11}, \ldots \} \)

- set builder notation
  \( S = \{ x \mid P(x) \} \) + the collection of elements in \( U \) for which \( P(x) \) is True.

If \( x_i \) is an element is \( S \)
we say \( x_i \in S \) / Note \( x_i \in U \)

* We want the collection of \( x_i \)'s for which \( P(x) \) is true, so only each \( x_i \) once.

Consider two sets, e.g. \( A = \{ x \mid P(x) \} \), \( B = \{ x \mid Q(x) \} \)
Intersection, union & difference.

Intersection: \( A \cap B \) (elements they share in common)

[Diagram of overlapping sets A and B with shading]

Union: \( A \cup B \) (all elements; counted only once)

\[ A \cup B = A + B - A \cap B \]

(each element only counted once)

(Inclusion/exclusion principle)

Difference: \( A - B \)

[Diagram of set A with the overlap with B shaded]

But set can be complex:

- Sets of sets; e.g., \( T = \{ \{ x_1, x_2 \}, \{ x_5, x_4, x_9 \} \ldots \} \)
- can have multiple variables from different \( U \)'s

E.g., \( T = \{ (x, y) \mid p(x) = T \land Q(y) = T \} \)
Very important sets

- The empty set $\emptyset = \{\}$
- The Natural numbers: \( \mathbb{N} = \{0, 1, 2, 3, \ldots\} \)
- The Integers: \( \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\} \)
- The positive Integers: \( \mathbb{Z}^+ = \{1, 2, 3, \ldots\} \)
- The real numbers \( \mathbb{R} \)

... onwards to Lec 4 slides...