Lec 11: Mid Term Review

- Review session Weds 5-6:30 in Kiebler #3.

- Propositions, \( p, q, r, t \), etc. \( \mapsto \{T, F\} \) map to \{1, 0\}.

- Compound props: operators \( \top, \land, \lor, \oplus \).

- Truth tables; length of table = \( 2^{\# \text{of propositions}} \)
  (easier than symbolic logic derivations)

- Implication \( p \rightarrow q \) or \( P(x) \rightarrow Q(x) \)
  Only false/invalid if \( q = F \) and \( p = T \)
  \( (p \rightarrow q) \equiv \neg p \lor q \)

  - Contrapositive \( q \rightarrow \neg p \equiv p \rightarrow \neg q \)

- Language: \( p \rightarrow q \) p is sufficient for q
  If p then q
  q if p
  q is necessary for p
  p only if q

- Biimplication \( (p \rightarrow q) \land (q \rightarrow p) =: p \leftrightarrow q \)
  "if and only if"
  "necessary and sufficient"
Logical equivalences (symbolic logic)

* See the tables on canvas Tables 6, 7, 8, Table 1 Sec. 1.3 Sec. 1.6.

De Morgan's:

\[ \neg (p \lor q) \equiv \neg p \land \neg q \]
\[ \neg (p \land q) \equiv \neg p \lor \neg q \]

(generalizes to multiple propositions
\[ \neg (p \lor q \lor r \lor \ldots) \]

- Predicate logic

\[ P(x) \]  
\[ \text{subject} \quad P(x, y), Q(x, y, z), \text{etc.} \]
\[ \text{predicate} \]

Once subject is given \( P(x) \rightarrow \) Proposition maps to \{T, F, \}

- Universal and existential quantification

(i.e. for which elements \( x \in U \) is \( P(x) = T \)?)

\[ \forall x \; \text{"for all } x\space\text{"} \]
\[ \exists x \; \text{"there exists an } x\space\text{"} \]

can specify the domain \( U \), e.g. \( \forall x \in \mathbb{R}, \exists x \in \mathbb{Z} \), etc.

De Morgan's

\[ \neg \forall x \; P(x) \equiv \exists x \; \neg P(x) \]
\[ \neg \exists x \; P(x) \equiv \forall x \; \neg P(x) \]
Build logical arguments w/ rules of inference.

\[
\begin{align*}
\text{Statement 1} & \quad \text{assumptions} \\
\text{Statement 2} & \quad \text{reasoning} \\
\vdots & \\
\text{conclusion}
\end{align*}
\]

Rules of inference

- **Modus Ponens (affirms)**
- **Modus Tollens (denies)**
- **Hypothetical syllogism**
  \[(p \rightarrow q) \land (q \rightarrow r) \rightarrow (p \rightarrow r)\]
- **Disjunctive syllogism (useful for simplifying compound props)**
  \[(p \lor q) \land \neg p \rightarrow q\]

Types of "trivial" proofs

\[(p \rightarrow q) \land \neg p \text{ then } "\text{vacuous}" \text{ proof.}\]

\[(p \rightarrow q) \land q \text{ then } "\text{trivial}" \text{ proof.}\]

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>p \rightarrow q</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>T</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>T</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>F</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>T</td>
</tr>
</tbody>
</table>
Fallacies:

Affirming the conclusion
saying \((p \rightarrow q) \land q\) therefore \(p\)

Denying the antecedent/premise
saying \((p \rightarrow q) \land \neg p\) therefore \(\neg q\)

Sets
select elements \(x \in U\).

Set builder notation

\[ A = \{x \mid P(x)\} \quad \text{(all } x \in U \text{ for which } P(x) = T) \]

\[ B = \{x, y \mid P(x, y)\} \]

\(2\) can be a complex instance (e.g., compound prop, implication, etc.)

Equivalence of sets

\[ A = B \iff \forall x \ (x \in A \iff x \in B) \]

Remember in a set each element occurs at most once; and the order does not matter.

Cardinality of a set \(|A| = \# \text{ of elements}\).

Empty set \(\emptyset = \{\}\)

Empty set is a valid subset of every set. (But: not a member of every set)
Subsets: \( A \subseteq C \) all \( a \in A \) are also \( a \in C \) but there is an \( c \in C \) with \( c \not\in A \)

\( A \subseteq C \) if \( A \) can equal \( C \).

A subset is all ways of choosing a set of elements from a given set.

Power set, \( P(A) = \text{set of all possible subsets of } A \).

\[ |P(A)| = 2^{\left|A\right|} \]

\( A = \{1, 2, 3\} \) or \( B = \{1, 2, 3, 3\} \)

\( \overbrace{\text{an element of a set can be a set.}}^\uparrow \)

\[ |B| = 3, \quad |P(B)| = 2^3 = 8 \]

\[ P(B) = \{\emptyset, \{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3, 3\}\} \]

- Venn diagrams
- Cartesian Products
- De Morgan's

\[ \overline{A \cup B} = A \cap \overline{B} \]

\[ \overline{A \cap B} = A \cup \overline{B} \]
- Union, intersection and subtraction of sets.

- Union of a collection of sets \( U \bigcup_{i=1}^{n} A_i \)

- Intersection \( \bigcap_{i=1}^{n} A_i \)

Functions \( f : A \rightarrow B \) (maps A to B)

Is a specific mapping:

- A function? - need every \( a \in A \) is mapped to one (not necessarily unique) \( b \in B \).

\[ \text{Domain} \quad A \quad \rightarrow \quad B \quad \text{Codomain} \]

- 1-to-1 (injective); a function where every \( a \in A \) maps to a unique \( b \in B \).
  (each \( b \in B \) has at most one partner in \( A \))
  \[ |\text{domain}| \leq |\text{codomain}| \]

- onto (surjective); a function where every \( b \in B \)
  has at least one partner in \( A \).
  \[ |\text{domain}| \geq |\text{codomain}| = |\text{range}| \]
Bijection is both 1-to-1 and onto.

- Every \( a \in A \) has a unique partner in \( B \).
- Every \( b \in B \) has a unique partner in \( A \).
- \( |\text{domain}| = |\text{codomain}| = |\text{range}| \)

* A bijection means \( f^{-1}(x) \) exists.

**Func. composition:**

\[
(f \circ g)(x) = f(g(x))
\]

Inverse func. composition \( (f \circ f^{-1})(x) = x \).

* Choice of domain & codomain can make all the difference
series: an order list, where repetition matters

Summations of sequences.

$$\sum_{k=0}^{n} a_k = a_0 + \sum_{k=1}^{n-1} a_k + a_n,$$

$$= a_0 + \sum_{j=1}^{n} a_j + \sum_{l=j+1}^{n} a_l$$

Imp. sums

$$\sum_{j=1}^{n} j, \sum_{j=1}^{n} j^2,$$ geometric series,

arithmetic series.

Proofs

Need some number theory background

- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{R}$, etc.

- Properties of these sets.
  - rational versus irrational
  - odd versus even
  - positive, negative, zero
  - prime versus composite.

(Needed these definitions as our axioms).
Methods of proof of an implication

- Direct proof $p \rightarrow q$
- Contrapositive $\neg q \rightarrow \neg p$
- Proof by contradiction
  - A proposition $P$; show $\neg p \rightarrow (r \land \neg r)$
    (where $p \rightarrow r$, typically a mathematical definition of property $P$).
  - Of implication $p \rightarrow q$
    Show $(p \land \neg q) \rightarrow (p \land \neg r) \lor (q \land \neg q)$

Argument

\[
\begin{align*}
  p & \text{ assert} \\
  \neg q & \text{ assert} \\
  \implies (p \land \neg q) \lor (q \land \neg q)
\end{align*}
\]

- Proof by cases
- Constructive existence proofs
- Non-constructive existence proofs.