The Theory of Sets

Rosen §2.1-2.2
Introduction to Set Theory

- A set is a new type of structure, representing an *unordered* collection (group, plurality) of zero or more *distinct* (different) objects.
- Set theory deals with operations between, relations among, and statements about sets.
- Sets are ubiquitous in computer software systems.
- *All* of mathematics can be defined in terms of some form of set theory (using predicate logic).
Basic notations for sets

• For sets, we’ll use variables $S$, $T$, $U$, …

• We can denote a set $S$ in writing by listing all of its elements in curly braces:
  – $\{a, b, c\}$ is the set of whatever 3 objects are denoted by $a$, $b$, $c$.

• *Set builder notation*: For any proposition $P(x)$ over any universe of discourse, $\{x | P(x)\}$ is *the set of all $x$ such that $P(x)$*. 

Basic properties of sets

- Sets are inherently unordered:
  - No matter what objects $a$, $b$, and $c$ denote,
    \[ \{a, b, c\} = \{a, c, b\} = \{b, a, c\} = \{b, c, a\} = \{c, a, b\} = \{c, b, a\}. \]

- All elements are distinct (unequal); multiple listings make no difference!
  - If $a=b$, then \( \{a, b, c\} = \{a, c\} = \{b, c\} = \{a, a, b, a, b, c, c, c, c\}. \)
  - This set contains (at most) 2 elements!
Definition of Set Equality

- Two sets are declared to be equal if and only if they contain exactly the same elements.
- In particular, it does not matter how the set is defined or denoted.
- For example: The set \( \{1, 2, 3, 4\} = \{x \mid x \text{ is an integer where } x>0 \text{ and } x<5 \} = \{x \mid x \text{ is a positive integer whose square is } >0 \text{ and } <25\} \)
Infinite Sets

• Conceptually, sets may be *infinite* (*i.e.*, not *finite*, without end, unending).

• Symbols for some special infinite sets:
  \[ \mathbb{N} = \{0, 1, 2, \ldots\} \quad \text{The Natural numbers.} \]
  \[ \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\} \quad \text{The Integers.} \]
  \[ \mathbb{R} = \text{The “Real” numbers, such as } 374.1828471929498181917281943125\ldots \]

• “Blackboard Bold” or double-struck font \( (\mathbb{N}, \mathbb{Z}, \mathbb{R}) \) is also often used for these special number sets.

• Infinite sets come in different sizes!
Venn Diagrams

John Venn
1834-1923
Venn Diagrams

Positive integers less than 10

John Venn
1834-1923
Venn Diagrams

John Venn
1834-1923

Even integers from 2 to 9
Basic Set Relations: Member of

• $x \in S$ ("$x$ is in $S$") is the proposition that object $x$ is an *element* or *member* of set $S$.
  - e.g. $3 \in \mathbb{N}$, “a” $\in \{x \mid x$ is a letter of the alphabet}$$
  -$Can define set equality in terms of $\in$ relation:
  \[
  \forall S, T: S = T \iff (\forall x: x \in S \iff x \in T)
  \]
  “Two sets are equal iff they have all the same members.”

• $x \notin S \equiv \neg (x \in S)$ “$x$ is not in $S$”
The Empty Set

- $\emptyset$ ("null", "the empty set") is the unique set that contains no elements whatsoever.
- $\emptyset = \{\} = \{x \mid \text{False}\}$
- No matter the domain of discourse, we have the axiom $\neg \exists x : x \in \emptyset$. 
Subset and Superset Relations

- \( S \subseteq T \) ("S is a subset of T") means that every element of S is also an element of T.
- \( S \subseteq T \iff \forall x (x \in S \rightarrow x \in T) \)
- \( \emptyset \subseteq S, S \subseteq S. \)
- \( S \supseteq T \) ("S is a superset of T") means \( T \subseteq S. \)
- Note \( S = T \iff S \subseteq T \land S \supseteq T. \)
- \( S \nsubseteq T \) means \( \neg (S \subseteq T) \), i.e. \( \exists x (x \in S \land x \notin T) \)
Proper (Strict) Subsets & Supersets

- \( S \subset T \) ("\( S \) is a proper subset of \( T \)") means that \( S \subseteq T \) but \( T \not\subseteq S \). Similar for \( S \supset T \).

Example:
\[
\{1,2\} \subset \{1,2,3\}
\]
Sets Are Objects, Too!

- The objects that are elements of a set may themselves be sets.
- E.g. let $S = \{ x \mid x \subseteq \{1,2,3\}\}$
  then $S = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$
- Note that $1 \neq \{1\} \neq \{\{1\}\}$

Very Important!
Cardinality and Finiteness

• $|S|$ (read “the cardinality of $S$”) is a measure of how many different elements $S$ has.

• E.g., $|\emptyset|=0$, $|\{1,2,3\}| = 3$, $|\{a,b\}| = 2$, $|\{\{1,2,3\},\{4,5\}\}| = ____$

• If $|S| \in \mathbb{N}$, then we say $S$ is finite. Otherwise, we say $S$ is infinite.

• What are some infinite sets we’ve seen?
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The *Power Set* Operation

- The *power set* \( P(S) \) of a set \( S \) is the set of all subsets of \( S \). \( P(S) \equiv \{ x \mid x \subseteq S \} \).
- *E.g.* \( P(\{a,b\}) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\} \).
- Sometimes \( P(S) \) is written \( 2^S \).
  Note that for finite \( S \), \( |P(S)| = 2^{|S|} \).
- It turns out \( \forall S: |P(S)| > |S| \), *e.g.* \( |P(\mathbb{N})| > |\mathbb{N}| \).
  *There are different sizes of infinite sets!*
Review: Set Notations So Far

- Variable objects $x, y, z$; sets $S, T, U$.
- Literal set \{a, b, c\} and set-builder \{x | P(x)\}.
- $\in$ relational operator, and the empty set $\emptyset$.
- Set relations $=, \subseteq, \supseteq, \subset, \supset, \notin$, etc.
- Venn diagrams.
- Cardinality $|S|$ and infinite sets $\mathbb{N}, \mathbb{Z}, \mathbb{R}$.
- Power sets $P(S)$. 
Naïve Set Theory is Inconsistent

- There are some naïve set *descriptions* that lead to pathological structures that are not *well-defined*. 
  - (That do not have self-consistent properties.)
- These “sets” mathematically *cannot* exist.
- *E.g.* let $S = \{ x \mid x \notin x \}$. Is $S \in S$?
- Therefore, consistent set theories must restrict the language that can be used to describe sets.
Ordered $n$-tuples

These are like sets, except that duplicates matter, and the order makes a difference.

For $n \in \mathbb{N}$, an ordered $n$-tuple or a sequence or list of length $n$ is written $(a_1, a_2, \ldots, a_n)$.

Its first element is $a_1$, etc.

Note that $(1, 2) \neq (2, 1)$.

Empty sequence, singlets, pairs, triples, quadruples, quintuples, ..., $n$-tuples.
Cartesian Products of Sets

• For sets $A$, $B$, their Cartesian product $A \times B \equiv \{(a, b) \mid a \in A \land b \in B\}$.

• E.g. $\{a,b\} \times \{1,2\} = \{(a,1),(a,2),(b,1),(b,2)\}$

• Note that for finite $A$, $B$, $|A \times B| = |A||B|$.

• Note that the Cartesian product is not commutative: i.e., $\neg \forall A B: A \times B = B \times A$.

• Extends to $A_1 \times A_2 \times \ldots \times A_n$...
Review

- Sets $S$, $T$, $U$... Special sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$.
- Set notations $\{a,b,...\}$, $\{x|P(x)\}$...
- Set relation operators $x \in S$, $S \subseteq T$, $S \supseteq T$, $S = T$, $S \subset T$, $S \supset T$. (These form propositions.)
- Finite vs. infinite sets.
- Set operations $|S|$, $P(S)$, $S \times T$.
- Next up: §1.5: More set ops: $\cup$, $\cap$, $\setminus$. 

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The Union Operator

- For sets $A$, $B$, their union $A \cup B$ is the set containing all elements that are either in $A$, or ("\lor") in $B$ (or, of course, in both).
- Formally, $\forall A,B: A \cup B = \{ x \mid x \in A \lor x \in B \}$.
- Note that $A \cup B$ is a superset of both $A$ and $B$ (in fact, it is the smallest such superset): $\forall A, B: (A \cup B \supseteq A) \land (A \cup B \supseteq B)$.
Union Examples

- \{a,b,c\} \cup \{2,3\} = \{a,b,c,2,3\}
- \{2,3,5\} \cup \{3,5,7\} = \{2,3,5,3,5,7\} = \{2,3,5,7\}
The Intersection Operator

- For sets $A$, $B$, their intersection $A \cap B$ is the set containing all elements that are simultaneously in $A$ and ("\&") in $B$.
- Formally, $\forall A, B: A \cap B = \{ x | x \in A \land x \in B \}$.
- Note that $A \cap B$ is a subset of both $A$ and $B$ (in fact it is the largest such subset): $\forall A, B: (A \cap B \subseteq A) \land (A \cap B \subseteq B)$.
Intersection Examples

- \( \{a,b,c\} \cap \{2,3\} = \_\_ \)
- \( \{2,4,6\} \cap \{3,4,5\} = \_\_\_\_\_ \)
Two sets $A$, $B$ are called disjoint (i.e., unjoined) iff their intersection is empty. ($A \cap B = \emptyset$)

Example: the set of even integers is disjoint with the set of odd integers.
Inclusion-Exclusion Principle

• How many elements are in $A \cup B$?
  $$|A \cup B| = |A| + |B| - |A \cap B|$$

• Example: How many students are on our class email list? Consider set $E = I \cup M$,
  $I = \{s \mid s$ turned in an information sheet$\}$
  $M = \{s \mid s$ sent the TAs their email address$\}$

• Some students did both!
  $$|E| = |I \cup M| = |I| + |M| - |I \cap M|$$
Set Difference

• For sets \( A, B \), the \textit{difference of }\( A \) \textit{and }\( B \), written \( A - B \), is the set of all elements that are in \( A \) but not \( B \). Formally:

\[
A - B \equiv \{ x \mid x \in A \land x \notin B \} = \{ x \mid \neg (x \in A \rightarrow x \in B) \}
\]

• Also called: 
The \textit{complement of }\( B \) \textit{with respect to }\( A \).
Set Difference Examples

- \( \{1,2,3,4,5,6\} - \{2,3,5,7,9,11\} = \)
  __________

- \( \mathbb{Z} - \mathbb{N} = \{\ldots, -1, 0, 1, 2, \ldots\} - \{0, 1, \ldots\} \)
  = \{x \mid x \text{ is an integer but not a nat. \#}\}
  = \{x \mid x \text{ is a negative integer}\}
  = \{\ldots, -3, -2, -1\} \)
Set Difference - Venn Diagram

- $A - B$ is what’s left after $B$ “takes a bite out of $A$”.

Set $A$

Set $B$
Set Difference - Venn Diagram

- $A \setminus B$ is what’s left after $B$ “takes a bite out of $A$”
Set Difference - Venn Diagram

- $A - B$ is what’s left after $B$ “takes a bite out of $A$”

Set Difference - Venn Diagram

- $A - B$ is what’s left after $B$ “takes a bite out of $A$”

Set $A - B$

Set $A$

Set $B$

Chomp!
Set Complements

- The *universe of discourse* can itself be considered a set, call it $U$.
- When the context clearly defines $U$, we say that for any set $A \subseteq U$, the *complement* of $A$, written $\overline{A}$, is the complement of $A$ w.r.t. $U$, i.e., it is $U-A$.
- *E.g.*, If $U=\mathbb{N}$, $\{3, 5\} = \{0,1,2,4,6,7,...\}$
More on Set Complements

- An equivalent definition, when \( U \) is clear:

\[
\overline{A} = \{ x \mid x \notin A \}
\]
Set Identities

- **Identity:** $A \cup \emptyset = A = A \cap U$
- **Domination:** $A \cup U = U$, $A \cap \emptyset = \emptyset$
- **Idempotent:** $A \cup A = A = A \cap A$
- **Double complement:** $(\overline{A}) = A$
- **Commutative:** $A \cup B = B \cup A$, $A \cap B = B \cap A$
- **Associative:** $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$
DeMorgan’s Law for Sets

• Exactly analogous to (and provable from) DeMorgan’s Law for propositions.

\[ A \cup B = \overline{A} \cap \overline{B} \]

\[ A \cap B = \overline{A} \cup \overline{B} \]
Proving Set Identities

To prove statements about sets, of the form $E_1 = E_2$ (where the $E$s are set expressions), here are three useful techniques:

1. Prove $E_1 \subseteq E_2$ and $E_2 \subseteq E_1$ separately.
2. Use set builder notation & logical equivalences.
3. Use a membership table.
Method 1: Mutual subsets

Example: Show $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

• Part 1: Show $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.
  – Assume $x \in A \cap (B \cup C)$, & show $x \in (A \cap B) \cup (A \cap C)$.
  – We know that $x \in A$, and either $x \in B$ or $x \in C$.
    • Case 1: $x \in B$. Then $x \in A \cap B$, so $x \in (A \cap B) \cup (A \cap C)$.
    • Case 2: $x \in C$. Then $x \in A \cap C$, so $x \in (A \cap B) \cup (A \cap C)$.
  – Therefore, $x \in (A \cap B) \cup (A \cap C)$.
  – Therefore, $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

• Part 2: Show $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. …
Review

• Sets $S$, $T$, $U$... Special sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$.
• Set notations $\{a,b,\ldots\}$, $\{x|P(x)\}$...
• Relations $x \in S$, $S \subseteq T$, $S \supseteq T$, $S = T$, $S \subset T$, $S \supset T$.
• Operations $|S|$, $P(S)$, $\times$, $\cup$, $\cap$, $\neg$, $\overline{S}$
• Set equality proof techniques:
  – Mutual subsets.
  – Derivation using logical equivalences.
Generalized Unions & Intersections

• Since union & intersection are commutative and associative, we can extend them from operating on ordered pairs of sets \((A, B)\) to operating on sequences of sets \((A_1, \ldots, A_n)\), or even on unordered sets of sets, \(X = \{A \mid P(A)\}\).
Generalized Union

• Binary union operator: \( A \cup B \)
• \( n \)-ary union:
  \[
  A \cup A_2 \cup \ldots \cup A_n \equiv ((\ldots((A_1 \cup A_2) \cup \ldots)\cup A_n)
  \]
  (grouping & order is irrelevant)
• “Big U” notation: \( \bigcup_{i=1}^{n} A_i \)
• Or for infinite sets of sets: \( \bigcup_{A \in X} A \)
Generalized Intersection

- Binary intersection operator: $A \cap B$
- $n$-ary intersection:
  $$A_1 \cap A_2 \cap \ldots \cap A_n \equiv (((A_1 \cap A_2) \cap \ldots) \cap A_n)$$
  (grouping & order is irrelevant)
- “Big Arch” notation:
  $$\bigcap_{i=1}^{n} A_i$$
- Or for infinite sets of sets:
  $$\bigcap_{A \in X} A$$