1. **Big-O notation.** (Bounded from above)
   Let f and g be functions from ($\mathbb{Z}$ or $\mathbb{R}$) to $\mathbb{R}$. We say that

$$f(x) = O(g(x))$$

   if there are constants c and k such that

$$|f(x)| \leq c|g(x)| \text{ whenever } x > k.$$

   Reads: "$f(x)$ is big-oh of $g(x)$".
   Means: beyond some point k, function $f(x)$ is at most a constant c times $g(x)$.
   Note: The **witnesses** c and k are not unique!
   In set notation: $\{f : \mathbb{R} \to \mathbb{R} | \ \exists ck > 0 \ \forall x > k \ |f(x)| \leq c|g(x)|\}$

2. Example:

   Theorem 1. *Let* $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, *where* $a_n, \ldots, a_0 \in \Re$. *Then* $f(x) = O(x^n)$.

   Proof. If $x > 1$,

$$\begin{aligned} |f(x)| &\leq |a_n|x^n + |a_{n-1}|x^{n-1} + \cdots + |a_1|x + |a_0| \\ &\leq (|a_n| + |a_{n-1}| + \cdots + |a_1| + |a_0|)x^n \\ &= cx^n. \end{aligned}$$

   □

3. Examples: Show that

   (a) $f(x) = x^2 + 2x + 1$ is $O(x^2)$.

   (b) $1 + 2 + 3 + \cdots + n$ is $O(n^2)$.

   (c) $f(n) = n!$ is $O(n^n)$.

   (d) $g(n) = \log n!$ is $O(n \log n)$.

4. The rules for function composition:

   Theorem 2. *If* $f_1(x) = O(g_1(x))$ *and* $f_2(x) = O(g_2(x))$, *then*

   - $(f_1 + f_2)(x) = O(\max(|g_1(x)|, |g_2(x)|))$, *and*

- $(f_1 \cdot f_2)(x) = O(g_1(x)g_2(x))$.

5. Example: Give a big-O notation esitmate for $f(n) = 3n \log n! + (n^2 + 3) \log n$.

6. **Big-Ω notation.** (Bounded from below)
   Let f and g be functions from ($\mathbb{Z}$ or $\mathbb{R}$) to $\mathbb{R}$. We say

$$f(x) = \Omega(g(x))$$

   if there are constants c and k such that

$$|f(x)| \geq c|g(x)| \text{ whenever } x > k.$$

   Reads: "$f(x)$ is big-omega of $g(x)$".
   Means: beyond some point k, function $f(x)$ is at least a constant c times $g(x)$.
   In set notation: $\{f : \mathbb{R} \to \mathbb{R} | \ \exists ck > 0 \ \forall x > k \ |f(x)| \geq c|g(x)|\}$
   E.g.: $f(x) = 8x^3 + 5x^2 + 6$ is $\Omega(x^3)$.

7. **Big-Θ notation.** (Exactly of the order)
   Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say

$$f(x) = \Theta(g(x))$$

   if

$$f(x) = O(g(x)) \text{ and } f(x) = \Omega(g(x)).$$

   Reads: "$f(x)$ is big-theta of $g(x)$" or "$f(x)$ is of **order** $g(x)$".
   In set notation: $\{f : \mathbb{R} \to \mathbb{R} | \ \exists c_1 c_2 k > 0 \ \forall x > k \ |c_1 g(x)| \leq |f(x)| \leq |c_2 g(x)|\}$
   E.g.: $1 + 2 + \cdots + n$ is of order n^2. That is,

$$1 + 2 + \cdots + n = \Theta(n^2).$$

8. Example:

   Theorem 3. *Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_n, \ldots, a_0 \in \Re$ and $a_n \neq 0$. Then $f(x)$ is of order x^n,* i.e.$f(x) = \Theta(x^n)$.

9. Summary table of notation

$f(x) \in$

$O(g(x))$	$\exists ck > 0\ \forall x > k\ \lvert f(x)\rvert \leq \lvert cg(x)\rvert$	For some $c > 0$, once $x > k$, $\lvert cg(x)\rvert$ dominates
$o(g(x))$	$\forall c > 0\ \exists k > 0\ \forall x > k\ \lvert f(x)\rvert \leq \lvert cg(x)\rvert$	For all $c > 0$, once $x > k$, $\lvert cg(x)\rvert$ dominates
$\Omega(g(x))$	$\exists ck > 0\ \forall x > k\ \lvert f(x)\rvert \geq \lvert cg(x)\rvert$	For some $c > 0$, once $x > k$, $\lvert cg(x)\rvert$ is smaller
$\omega(g(x))$	$\forall c > 0\ \exists k > 0\ \forall x > k\ \lvert f(x)\rvert \geq \lvert cg(x)\rvert$	For all $c > 0$, once $x > k$, $\lvert cg(x)\rvert$ is smaller
$\Theta(g(x))$	$\exists c_1 c_2 k > 0\ \forall x > k\ \lvert c_1 g(x)\rvert \leq \lvert f(x)\rvert \leq \lvert c_2 g(x)\rvert$	For some $c_1, c_2 > 0$, once $x > k$, $f(x)$ and $g(x)$ are similar