1. **Big-$O$ notation.** (Bounded from above)
   Let $f$ and $g$ be functions from ($\mathbb{Z}$ or $\mathbb{R}$) to $\mathbb{R}$. We say that
   
   \[
   f(x) = O(g(x))
   \]

   if there are constants $c$ and $k$ such that
   
   \[
   |f(x)| \leq c|g(x)| \text{ whenever } x > k.
   \]

   Reads: “$f(x)$ is big-oh of $g(x)$”.
   Means: beyond some point $k$, function $f(x)$ is at most a constant $c$ times $g(x)$.
   Note: The witnesses $c$ and $k$ are not unique!

   In set notation: \[
   \{ f : \mathbb{R} \to \mathbb{R} \mid \exists c, k > 0 \forall x > k \ |f(x)| \leq c|g(x)| \}
   \]

2. **Example:**
   
   **Theorem 1.** Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where $a_n, \ldots, a_0 \in \mathbb{R}$. Then $f(x) = O(x^n)$.
   
   **Proof.** If $x > 1$,
   
   \[
   |f(x)| \leq |a_n| x^n + |a_{n-1}| x^{n-1} + \cdots + |a_1| x + |a_0| \\
   \leq (|a_n| + |a_{n-1}| + \cdots + |a_1| + |a_0|) x^n \\
   = cx^n.
   \]

3. **Examples:** Show that
   
   (a) $f(x) = x^2 + 2x + 1$ is $O(x^2)$.
   (b) $1 + 2 + 3 + \cdots + n$ is $O(n^2)$.
   (c) $f(n) = n!$ is $O(n^n)$.
   (d) $g(n) = \log n!$ is $O(n \log n)$.

4. **The rules for function composition:**
   
   **Theorem 2.** If $f_1(x) = O(g_1(x))$ and $f_2(x) = O(g_2(x))$, then
   
   \[
   (f_1 + f_2)(x) = O(\max(|g_1(x)|, |g_2(x)|)), \text{ and}
   \]

\[
\]
5. Example: Give a big-$O$ notation estimate for $f(n) = 3n \log n! + (n^2 + 3) \log n$.

6. **Big-$\Omega$ notation.** (Bounded from below)
   Let $f$ and $g$ be functions from $(\mathbb{Z} \text{ or } \mathbb{R})$ to $\mathbb{R}$. We say
   \[ f(x) = \Omega(g(x)) \]
   if there are constants $c$ and $k$ such that
   \[ |f(x)| \geq c|g(x)| \text{ whenever } x > k. \]
   Reads: “$f(x)$ is big-omega of $g(x)$”.
   Means: beyond some point $k$, function $f(x)$ is at least a constant $c$ times $g(x)$.
   In set notation: \{ $f : \mathbb{R} \to \mathbb{R} | \exists c > 0 \forall x > k |f(x)| \geq c|g(x)|$ \}
   E.g.: $f(x) = 8x^3 + 5x^2 + 6$ is $\Omega(x^3)$.

7. **Big-$\Theta$ notation.** (Exactly of the order)
   Let $f$ and $g$ be functions from the set of integers or the set of real numbers to the set of real numbers. We say
   \[ f(x) = \Theta(g(x)) \]
   if
   \[ f(x) = O(g(x)) \text{ and } f(x) = \Omega(g(x)). \]
   Reads: “$f(x)$ is big-theta of $g(x)$” or “$f(x)$ is of order $g(x)$”.
   In set notation: \{ $f : \mathbb{R} \to \mathbb{R} | \exists c_1, c_2 > 0 \forall x > k |f(x)| \leq |c_1g(x)| \leq |c_2g(x)|$ \}
   E.g.: $1 + 2 + \cdots + n$ is of order $n^2$. That is,
   \[ 1 + 2 + \cdots + n = \Theta(n^2). \]

8. Example:

   **Theorem 3.** Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where $a_n, ..., a_0 \in \mathbb{R}$ and $a_n \neq 0$. Then $f(x)$ is of order $x^n$, i.e. $f(x) = \Theta(x^n)$. 

9. Summary table of notation

\[ f(x) \in \]

| \( O(g(x)) \)       | \( \exists c k > 0 \forall x > k |f(x)| \leq |cg(x)| \) | For some \( c > 0 \), once \( x > k, |cg(x)| \) dominates |
|----------------------|--------------------------------------------------|-------------------------------------------------|
| \( o(g(x)) \)       | \( \forall c > 0 \exists k > 0 \forall x > k |f(x)| \leq |cg(x)| \) | For all \( c > 0 \), once \( x > k, |cg(x)| \) dominates |
| \( \Omega(g(x)) \)  | \( \exists c k > 0 \forall x > k |f(x)| \geq |cg(x)| \) | For some \( c > 0 \), once \( x > k, |cg(x)| \) is smaller |
| \( \omega(g(x)) \)  | \( \forall c > 0 \exists k > 0 \forall x > k |f(x)| \geq |cg(x)| \) | For all \( c > 0 \), once \( x > k, |cg(x)| \) is smaller |
| \( \Theta(g(x)) \)  | \( \exists c_1 c_2 k > 0 \forall x > k |c_1 g(x)| \leq |f(x)| \leq |c_2 g(x)| \) | For some \( c_1, c_2 > 0 \), once \( x > k, f(x) \) and \( g(x) \) are similar |