1. **Direct proof.**
   The implication \( p \rightarrow q \) can be proved by showing that if \( p \) is true then \( q \) must also be true. A proof of this kind is called a *direct proof*.

2. **Indirect Proof.**
   **Proof by contraposition:** Since the implication \( p \rightarrow q \) is equivalent to its contrapositive, \( \neg q \rightarrow \neg p \), the implication \( p \rightarrow q \) can be proved by showing that \( \neg q \rightarrow \neg p \) is true. This related implication is usually proved directly. An argument of this type is called an *indirect proof*.
   Example: Prove “If \( 3n + 2 \) is odd, then \( n \) is odd”.
   A *vacuous proof* is established by showing \( \neg p \).
   A *trivial proof* is established by showing \( q \) is true.

3. **Proof by contradiction.**
   (a) *For proposition \( p \):* Assume \( \neg p \) is true and show this leads to both \( r \) and \( \neg r \) for some independent proposition \( r \); in other words \( \neg p \rightarrow (r \land \neg r) \).
   (b) *For implication \( p \rightarrow q \):* By assuming that the hypothesis \( p \) is true and that the conclusion \( q \) is false, then using \( p \) and \( \neg q \) as well as other axioms, definitions, and previously derived theorems, derives a contradiction. Proofs are based on noting that
   \[
   (((p \rightarrow q) \land p) \land \neg q) \equiv (q \land \neg q) \quad \text{likewise} \quad (p \land (\neg q \rightarrow \neg p)) \equiv (p \land \neg p).
   \]
   Exmpl (a): Prove that \( \sqrt{2} \) is irrational.
   Exmpl (b): Prove that for all real numbers \( x \) and \( y \), if \( x + y \geq 2 \), then either \( x \geq 1 \) or \( y \geq 1 \).

4. **Equivalence proof** (or “if-and-only-if proof”, “necessary-and-sufficient proof”).
   To prove a theorem that is an equivalence, i.e., \( p \leftrightarrow q \), the tautology
   \[
   (p \leftrightarrow q) \leftrightarrow ((p \rightarrow q) \land (q \rightarrow p))
   \]
   can be used. That is, the proposition “\( p \) if and only if \( q \)” can be proved if both the implication “if \( p \) then \( q \)” and “if \( q \) then \( p \)” are proved.
   Example: Prove the theorem: The integer \( n \) is odd if and only if \( n^2 \) is odd.”
5. **Exhaustive proof / proof by cases**

*Exhaustive proof:* Proof by showing it holds for all possible \( x \) in \( U \) (e.g. a truth table).

*Proof by cases:* Proof by showing it holds for all possible cases. (Useful when direct proof not simple but the extra information in the cases let’s you move forward.)

Example: Prove that if \( n \) is an integer, then \( n^2 \geq n \). (Three cases, \( n < 0, n = 0, n > 0 \).)

*Without loss of generality (WLOG):* Same proof holds for all cases. Quite useful but gets one into trouble (a common mistake in a proof).

6. **Constructive existence proof:** of statement of the form \( \exists x P(x) \). Just find one value of \( x \) in \( U \) for which \( P(x) \) is true. (Hence “constructive”)

Example: Prove there exists a positive integer \( n \) that is the sum of cubes of positive integers in two ways: \( \exists n : n = i^3 + j^3 = k^3 + l^3 \).

7. **Nonconstructive existence proof.** Don’t pinpoint the exact values that satisfy, just show they must exist.

Example: Show there exist irrational numbers \( x \) and \( y \) such that \( x^y \) is rational. In propositional language \( \exists x, y ((\text{Irrational}(x) \land \text{Irrational}(y)) \to \text{Rational}(x^y)) \).

Example: Prove there are infinitely many primes; \( \forall n \exists p > n \) where \( p \) is prime.

8. **Proof by counterexample.**

The goal of such a proof is to show \( \forall x P(x) \) is false.

(Note: Showing \( \exists x P(x) \) is false is counterexample for \( \forall x P(x) \), but this is not a counterexample for the conjecture \( \exists x P(x) \).)

Example: Show that the assertion “All primes are odd” is false.

9. **Mistakes in proofs.**

False premises

Circular reasoning

Not considering all cases (e.g. missing \( x = 0 \) case).

Showing \( \exists \) a satisfying value to prove \( \forall \) statements.

Assuming \( \exists \) means that all instances satisfy.

Example: What is wrong with the famous supposed “proof” that \( 1 = 2 \)?