1. A **permutation** of a set of distinct objects is an *ordered* arrangement of these objects. An *ordered* arrangement of r elements of a set is called an r**-permutation**.

   $P(n, r) =$ the number of r-permutations of a set with n elements.

   Theorem: $P(n, r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$

   Examples:

   (a) In how many ways can we select a chair, vice-chair, treasurer, secretary from a group of 10 people?

   (b) How many permutations of the letters $ABCDEFGH$ contain the string ABC?

   (c) Traveling salesperson problem.

2. A r**-combination** counts the number of ways r elements can be selected from a set A with $|A| \geq r$. It is an *unordered* selection of r elements from the set.

   $C(n, r) =$ the number of r-combinations of a set with n distinct elements.
   $C(n, r)$ is usually written as $\left(\begin{array}{c} n \\ r \end{array} \right)$ which stated in English is "n choose r".

   Theorem: $C(n, r) = \left(\begin{array}{c} n \\ r \end{array} \right) = \frac{P(n,r)}{P(r,r)} = \frac{n!}{r!(n-r)!}$

   Proof: $P(n, r) = C(n, r) \cdot P(r, r)$.
   Note: $C(n, r) = C(n, (n-r))$.

   Example:

   (a) How many poker hands of five cards can be dealt from a 52 card deck?

   (b) How many ways can we select a committee of two women and three men from a group of 5 women and 6 men? (Note: Consider combinations for women, then for men. Do we use sum rule or product rule?)

   Items 3-8 below are FYI and not on exam:

3. The Binomial Theorem: Let x and y be variables, and let n be a positive integer. Then

$$(x + y)^n = \sum_{j=0}^{n} C(n, j) x^{n-j} y^j.$$

Therefore, $C(n,r)$ is also called a **binomial coefficient**.

Proof: Use a **combinatorial proof** (much easier than using mathematical induction).

Example: a) What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$?

b) What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

4. A **combinatorial proof** of an identity uses counting arguments to show both sides of the identity count the same objects but in different ways.

5. Theorem:
$$\sum_{j=0}^{n} C(n,j) = 2^n. \tag{1}$$
Direct proof: Binomial theorem with $x = 1$ and $y = 1$.
Combinatorial proof: Counting number of subsets.

6. Pascal's Identity:
$$C(n+1,k) = C(n,k-1) + C(n,k).$$
Proof:
   - Method 1: use algebraic proof.
   - Method 2: use a combinatorial proof.

7. Pascal's triangle.

8. The identity (1) shows that the row sum of the Pascals triangle is 2^n.

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$

$$\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7}$$

$$\binom{8}{0} \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8}$$

$$\cdots$$

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$1$$

$$1 \quad 1$$

$$1 \quad 2 \quad 1$$

$$1 \quad 3 \quad 3 \quad 1$$

$$1 \quad 4 \quad 6 \quad 4 \quad 1$$

$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$

$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$

$$1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1$$

$$1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1$$

$$\cdots$$