1. The **product rule**: Suppose that a procedure can be broken into a sequence of two tasks. If there are \( n_1 \) ways to do the first task, and for each of the ways of doing the first task, there are \( n_2 \) ways to do the second task, then there are \( n_1 n_2 \) ways to do the procedure.

In set notation: Let \( A_1 \) be the set of the \( n_1 \) ways to do the first task, and \( A_2 \) the set of \( n_2 \) ways to do the second task. Then

\[
|A_1 \times A_2| = |A_1| \times |A_2| = n_1 n_2.
\]

2. The **extended product rule**: \( |A_1 \times A_2 \times \cdots \times A_m| = |A_1| \times |A_2| \times \cdots \times |A_m| \).

**Examples**:

(a) How many numbers in the range 1000 – 9999?

(b) How many numbers in the range 1000 – 9999 do not have any repeated digits?

(c) How many one-to-one functions are there from a set with \( m \) elements to a set with \( n \) elements?

(d) Recall cardinality of power set of a set \( A \): \( |P(A)| = 2^{|A|} \) (the number of subsets).

(e) How many 10-digit telephone numbers of the form \( NXX-NXX-XXXX \), where \( N \in \{2, 3, \ldots, 9\} \) and \( X \in \{0, 1, \ldots, 9\} \)?

3. The **sum rule**: If a task can be done either in one of \( n_1 \) ways or in one of \( n_2 \) ways, where none of the set of \( n_1 \) ways are the same as any of the \( n_2 \) ways, then there are \( n_1 + n_2 \) ways to do the task.

In set notation: Let \( A_1 \) be the set of the first \( n_1 \) ways to do a task, and \( A_2 \) the set of the second \( n_2 \) ways to do the task. Assume that \( A_1 \) and \( A_2 \) are disjoint, i.e., \( A_1 \cap A_2 = \emptyset \), then the set of ways to do the task is \( A_1 \cup A_2 \) and the number of ways is

\[
|A_1 \cup A_2| = |A_1| + |A_2| = n_1 + n_2.
\]

4. The **extended sum rule**: \( |A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m| \) provided that \( A_i \cap A_j = \emptyset \) whenever \( i \neq j \).

**Example**: How many ways can we choose one student from the CS dept if there are \( f \) freshman, \( p \) sophomores, \( j \) juniors and \( s \) seniors?
5. The **inclusion-exclusion rule**: Suppose that a task can be done in \( n_1 \) or \( n_2 \) ways, but that some of the set of \( n_1 \) ways to do the task are the same as some of the \( n_2 \) ways. First add \( n_1 \) and \( n_2 \) ways, then subtract the number of common ways.

In set notation:
\[
|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.
\]

**Example**: How many bit strings of length eight start with a 1 bit or end with two bits 00?

6. **More complex counting problems**: many counting problems cannot be solved using just the sum rule or the product rule, but can be solved using both of these rules in combination.

**Examples**:

(a) Let \( P_6 \) denote the number of passwords of six characters, where each character is an uppercase letter or digit, and each password must contain at least one digit. What’s \( P_6 \)?

(b) How many such passwords are there of length six to eight characters long?

(c) Number of IPv4 Internet addresses.

7. Tying it all together – e.g., problem 5.1.21.

8. **The Pigeonhole Principle**: If \( k + 1 \) or more objects are placed into \( k \) boxes, then there is at least one box containing two or more of the objects.

**Example**: Among any group of 367 people, there must be at least two people with the same birthday, because there are only 366 possible birthdays.