1. The **product rule**: Suppose that a procedure can be broken into a sequence of two tasks. If there are \( n_1 \) ways to do the first task, and for each of the ways of doing the first task, there are \( n_2 \) ways to do the second task, then there are \( n_1 n_2 \) ways to do the procedure.

   In set notation: Let \( A_1 \) be the set of the \( n_1 \) ways to do the first task, and \( A_2 \) the set of \( n_2 \) ways to do the second task. Then

   \[
   |A_1 \times A_2| = |A_1| \times |A_2| = n_1 n_2.
   \]

2. The **extended product rule**: \( |A_1 \times A_2 \times \cdots \times A_m| = |A_1| \times |A_2| \times \cdots \times |A_m| \).

   **Examples**:
   
   (a) How many numbers in the range 1000 – 9999?
   
   (b) How many numbers in the range 1000 – 9999 do not have any repeated digits?
   
   (c) How many one-to-one functions are there from a set with \( m \) elements to a set with \( n \) elements?
   
   (d) Recall cardinality of power set of a set \( A \): \( |P(A)| = 2^{|A|} \) (the number of subsets).
   
   (e) How many 10-digit telephone numbers of the form \( NXX-NXX-XXXX \), where \( N \in \{2, 3, \ldots, 9\} \) and \( X \in \{0, 1, \ldots, 9\} \)?

3. The **sum rule**: If a task can be done either in one of \( n_1 \) ways or in one of \( n_2 \) ways, where none of the set of \( n_1 \) ways are the same as any of the \( n_2 \) ways, then there are \( n_1 + n_2 \) ways to do the task.

   In set notation: Let \( A_1 \) be the set of the first \( n_1 \) ways to do a task, and \( A_2 \) the set of the second \( n_2 \) ways to do the task. Assume that \( A_1 \) and \( A_2 \) are disjoint, i.e., \( A_1 \cap A_2 = \emptyset \), then the set of ways to do the task is \( A_1 \cup A_2 \) and the number of ways is

   \[
   |A_1 \cup A_2| = |A_1| + |A_2| = n_1 + n_2.
   \]

4. The **extended sum rule**: \( |A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m| \) provided that \( A_i \cap A_j = \emptyset \) whenever \( i \neq j \).

   **Example**: How many ways can we choose one student from the CS dept if there are \( f \) freshman, \( p \) sophomores, \( j \) juniors and \( s \) seniors?
5. The **inclusion-exclusion rule**: Suppose that a task can be done in \( n_1 \) or \( n_2 \) ways, but that some of the set of \( n_1 \) ways to do the task are the same as some of the \( n_2 \) ways. First add \( n_1 \) and \( n_2 \) ways, then subtract the number of common ways.

In set notation:

\[
|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.
\]

*Example*: How many bit strings of length eight start with a 1 bit or end with two bits 00?

6. **More complex counting problems**: Many counting problems cannot be solved using just the sum rule or the product rule, but can be solved using both of these rules in combination. **Examples**:  
   (a) Let \( P_6 \) denote the number of passwords of six characters, where each character is an uppercase letter or digit, and each password must contain at least one digit. What’s \( P_6 \)?  
   (b) How many such passwords are there of length six to eight characters long?  
   (c) Number of IPv4 Internet addresses (see page 341 in Rosen).

7. Tying it all together – e.g., problem 5.1.21.

8. **The Pigeonhole Principle**: If \( k + 1 \) or more objects are placed into \( k \) boxes, then there is at least one box containing two or more of the objects. **Example**: Among any group of 367 people, there must be at least two people with the same birthday, because there are only 366 possible birthdays.

9. **The Generalized Pigeonhole Principle**: If \( N \) objects are placed into \( k \) boxes, then there is at least one box containing at least \( \lceil N/K \rceil \) objects. **Examples**:  
   (a) Among 100 people there are at least \( \lceil 100/12 \rceil = 9 \) who were born in the same month.  
   (b) Example: What is the least number of area codes needed to guarantee that the 25 million phones in a state have distinct 10-digits telephone numbers \( NXX-NXX-XXX \) where \( N \in \{2, 3, \ldots, 9\} \) and \( X \in \{1, 2, \ldots, 9\} \)?

10. An elegant application of the pigeonhole principle. **Example**: During a month of 30 days a baseball team plays at least 1 game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.