1. **Recursion** is the general term for defining an object in terms of itself.
   e.g., An *inductive proof* establishes the truth of $P(k + 1)$ recursively in terms of $P(k)$.
   e.g., A fractal is a recursive object (see the Koch snowflake).

2. A **recursive definition** of a function, predicate, set, or other structure, defines the larger elements in terms of the smaller elements.
   Example: Recall an arithmetic progression: $\{a_j\}$ where $a_j = b + jd$. We can define this series by recursion where $a_0 = b$ and $a_j = a_{j-1} + d$. (For instance $\{1, 3, 5, 7, 9, \ldots\}$.)

3. A function $f(n)$ where $n$ is a nonnegative integer is **defined recursively** if
   
   (a) $f(0), f(1), \ldots, f(k)$ are given, and
   
   (b) for $n > k$, a rule is given for computing $f(n)$ from $f(0), f(1), \ldots, f(n - 1)$.

   Example: $f(n) = n!$. Define as $f(0) = 1$ (base case) and $f(n) = nf(n - 1)$ (recursion).

4. The Fibonacci numbers $f_0, f_1, f_2, \ldots$ are defined by the following:

   $$f_n = \begin{cases} 
   0 & \text{if } n = 0 \
   1 & \text{if } n = 1 \
   f_{n-1} + f_{n-2} & \text{if } n \geq 2
   \end{cases}$$

   $$\lim_{n \to \infty} \left( \frac{f_{n+1}}{f_n} \right) = \phi$$ where $\phi = \frac{1+\sqrt{5}}{2} \approx 1.6180327868852$ (“the golden ratio”).

5. A **recursive algorithm** solves a problem by reducing it to an instance of the same problem with a smaller input (“Top down”). A recursive algorithm calls itself.
   
   a) Algorithm 1 recursively computes the value of $a^n$.
   
   b) Algorithm 2 recursively computes the value of $n!$.

6. An **iterative algorithm** starts with the value of the function at one (or more) base case(s), and successively applies the recursive definition to build up the values of the function at successively larger integers (“Bottom up”). An iterative algorithm typically uses *for* loops.
   
   a) Algorithm 3 iteratively computes the value of $n!$.

7. **Binary search**: Algorithm 4 is a recursive, whereas Alg 5 is original iterative version.

8. Algorithm 6 is an algorithm for computing the $n$th Fibonacci number.
   Is it *recursive* or *iterative*?
Algorithm 1 A recursive algorithm for computing $a^n$.

```plaintext
POWER (a, n)
    if $n = 0$ then
        POWER(a,n)=1
    else
        POWER(a,n) = a·POWER(a, n - 1)
    end if
```

Algorithm 2 A recursive algorithm for computing $n!$.

```plaintext
FACTORIAL (n)
    if $n = 0$ then
        FACTORIAL(n) =1
    else
        FACTORIAL(n) = n·FACTORIAL(n - 1)
    end if
```

Algorithm 3 An iterative algorithm for computing $n!$.

```plaintext
FACTORIAL (n)
    x ← 1
    for $i = 2, 3, \ldots, n$ do
        x ← x·i
    end for
    return x
```

Algorithm 4 A recursive binary search algorithm. $x, i, j$ are integers with $1 \geq i, j \geq n$.

```plaintext
BINARYSEARCH (x, i, j)
    m ← ⌊(i + j)/2⌋
    if $x = a_m$ then
        return m
    else if $x < a_m$ and $i < m$ then
        return BINARYSEARCH(x, i, m - 1)
    else if $x > a_m$ and $j > m$ then
        return BINARYSEARCH(x, m + 1, j)
    else
        return 0
    end if
```
Algorithm 5 The original binary search algorithm from Handout 7.

**BINARY-SEARCH** \((a, x)\)

\[
i \leftarrow 1 \{i \text{ is the left endpoint of search interval}\}
\]

\[
j \leftarrow n \{j \text{ is the right endpoint of search interval}\}
\]

\[
\text{while } i < j \text{ do}
\]

\[
m \leftarrow \lfloor (i + j)/2 \rfloor
\]

\[
\text{if } x > a_m \text{ then}
\]

\[
i \leftarrow m + 1
\]

\[
\text{else}
\]

\[
j \leftarrow m
\]

\[
\text{end if}
\]

\[
\text{end while}
\]

\[
\text{if } x = a_i \text{ then}
\]

\[
\text{location } \leftarrow i
\]

\[
\text{else}
\]

\[
\text{location } \leftarrow 0
\]

\[
\text{end if}
\]

\[
\text{return } \text{location} \{\text{location is the subscript of the term that equals } x, \text{ or is 0 if } x \text{ is not found}\}
\]

Algorithm 6 An algorithm for computing the \(n\)th Fibonacci number.

**FIBONACCI** \((n)\)

\[
\text{if } n = 0 \text{ then}
\]

\[
y \leftarrow 0
\]

\[
\text{else}
\]

\[
x \leftarrow 0, y \leftarrow 1
\]

\[
\text{for } i = 1, 2, \ldots, n - 1 \text{ do}
\]

\[
z \leftarrow x + y
\]

\[
x \leftarrow y
\]

\[
y \leftarrow z
\]

\[
\text{end for}
\]

\[
\text{end if}
\]

\[
\text{return } y
\]
9. Note *recursion* is often less efficient than *iteration*! But it is easier to implement and elegant.

![Fig 1. First four iterations in constructing the Koch snowflake.](image1)

![Fig 2. Fibonacci spiral out to squares of size 34.](image2)