“Exploring network robustness”
Recall: Preferential Attachment random graphs: The Barabási and Albert model

- A discrete time process.
- Start with a single isolated node.
- At each time step, a new node arrives.
- This node makes \( m \) connections to already existing nodes. (Why \( m \) edges?)
- We are interested in the limit of large graph size.
Probability

- Probability incoming node attaches to node $j$:

$$Pr(t + 1 \rightarrow j) = \frac{d_j}{\sum_j d_j}.$$  

- Probability incoming node attaches to any node of degree $k$:

$$\frac{\# \text{ nodes of degree } k}{\# \text{ nodes}} \times \frac{\text{degree of that node}}{\text{degree sum over all nodes}} =$$

$$\frac{kp_k}{\sum_k d_k} = \frac{kp_k}{2mn}$$
Network evolution
Process on the degree sequence

- Note that $p_k$ will change in time!
  So we show denote this explicitly: $p_{k,t}$

- Also, when a node of degree $k$ gains an attachment, it becomes a node of degree $k + 1$.

- When the new node arrives, it increases by one the number of nodes of degree $m$. 
Recursion for $p_m$

\[
p_k = \frac{(k-1)(k-2)\cdots(m)}{(k+2)(k+1)\cdots(m+3)} \cdot p_m = \frac{m(m+1)(m+2)}{(k+2)(k+1)k} \cdot \frac{2}{(m+1)}
\]

\[
p_k = \frac{2m(m+1)}{(k+2)(k+1)k}
\]

For $k \gg 1$

\[
p_k \sim k^{-3}
\]
Simulating PA

Basic code for simulating PA with $m = 1$ using R:

- `runPA ← function(N=100)`

```r
{  
  # outLink[i] is the parent of i  
  outLink ← numeric(N)  
  # numlinks[i] is number total-links (in and out) for node i  
  numLinks ← numeric(N)+1  
  for(i in 2:N)
    {
      p ← sample(c(1:(i-1)), size=1, prob=numLinks[1:(i-1)])  
      outLink[i] ← p  
      numLinks[p] ← numLinks[p]+1  
    }
  return(list(outLink, numLinks))
}
```
Visualizing a PA graph \((m = 1)\) at \(n = 5000\)
Exploring PA networks

Demo:

- R
- Graphviz

We will use these as tools to explore Network robustness (defined here as maintaining connectivity despite node and edge deletion).

- Rest of lecture will be interactive demo....
Robustness of power law random graphs to node and edge deletion

- Extremely resilient to random failure.
- Extremely sensitive to targeted attack.

The “achilles’ heel” of the Internet