Community Structure and Beyond

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Why do we care about community structure?
Large Networks
Discussion Outline

• Overview of past work on community structure.

• How to determine the “best” number of communities.

• Fast linear algebra based method.

• Bringing in statistics.
A Brief History of Methods

- **Spectral methods**, graph partitioning problems.
  - A well known example is **spectral bisection**, which uses the graph/network **Laplacian**.
    
    \[ L_{ij} = \delta_{ij}k_i - A_{ij} \]

- In the special case of a network having only two communities, Fiedler proposed a method for identifying the members nodes.
A Brief History of Methods

- **Hierarchical clustering**: groups nodes into communities such that nodes within a community are *similar* to each other in some sense; widely used in sociology.

- Technique 1) calculate a weight, $W_{ij}$ for every pair of nodes in the network 2) then take the $n$ nodes with no edges between them an add edges between pairs one by one in order of their weights, from strongest to weakest.

- Many ways exist for calculating the $W_{ij}$ values.

- The entire process is frequently represented as a **dendrogram**, a visualization of the vertices coalescing into communities.
A Brief History of Methods
GN Algorithm

- **Girvan-Newman Algorithm**: a divisive method for determining community structure that focuses on the *betweenness* of edges.

- **Edge betweenness**: the number of shortest paths between pairs of vertices that run along an edge.

- Removing edges of high betweenness breaks up the connected network into communities.


**Algorithm**

1. Calculate the betweenness for all edges in the network.
2. Remove the edge with the highest betweenness.
3. Recalculate betweenness for all edges affected by the removal.
4. Repeat from steps 2 until no edges remain.
GN Algorithm
GN Algorithm

The classic "Karate Club" example
Modularity


- Introduced by Newman and Girvan to quantify which division of a network into communities/groups was the best.


- **Modularity**: the fraction of edges falling within communities minus the expected fraction of such edges.

\[
e_{ij} : \text{the fraction of all edges in the network that link vertices in community } i \text{ to vertices in community } j.
\]

\[
a_i = \sum_j e_{ij} : \text{the fraction of edges that connect to vertices in community } i.
\]

\[
Q = \sum (e_{ii} - a_i^2) = \text{Tr} e - \|e^2\|
\]
Modularity


Again the “Karate Club”
Modularity
We love to study ourselves...
A New-New Approach

- Newman later returned to the subject of community structure and modularity with a new-new approach.

- Modularity maximization was the leading tool for determining optimal community structure.

  - Simulated annealing had been shown to be very successful, but slow.

Calculating Modularity

Rewrite modularity using the adjacency matrix.

\[ Q = \frac{1}{2m} \sum_{i,j=1}^{n} [A_{ij} - P_{ij}] \delta_{c_i, c_j} \]

\[ A_{ij} = \begin{cases} 1, & \text{if there is an edge from } j \text{ to } i \\ 0, & \text{otherwise} \end{cases} \]

\[ P_{ij} = \text{the expected number of edges from } j \text{ to } i. \]

\[ c_i = \text{the community to which } i \text{ belongs.} \]

**How do we determine the “expected” number of edges between two vertices?**

\[ P_{ij} = \frac{k_i k_j}{2m} \]
Division of a Network into Two Communities

\[ Q = \frac{1}{4m} \sum_{ij} \left( A_{ij} - \frac{k_i k_j}{2m} \right) s_i s_j = \frac{1}{4m} s^T B s, \]

\[ s = \sum_{t=1}^{n} a_t u_t \text{ with } a_t = u_t^T \cdot s. \]

\[ Q = \sum_{i} a_i u_i^T B \sum_{j} a_j u_j = \sum_{t=1}^{n} (u_t^T \cdot s)^2 \beta_t, \]
Again with the “Karate Club”
Communities with Edge Direction Bias
Communities with Edge Direction Bias
Edge Direction Bias in Real Networks
Edge Direction Bias in Real Networks

Purdue

Northwestern
Pennsylvania State

Ohio State

Illinois

Michigan State

Minnesota

Wisconsin

Iowa

Purdue

Indiana

Michigan
Exploratory Analysis of Structure in Networks

- Previously we identified communities in networks because we specifically sought a method to detect modules in networks.

- Reliance on specific measures of network structure where we are required to know the type of structure, for which we are looking, in advance can be limiting.

- We turn to probabilistic techniques and the Expectation Maximization (EM) Algorithm to identify general patterns of connection between vertices.
The Method

There are three types of quantities in this method of approach.

1. Observed data: the actual edges falling between pairs of vertices in a network ($\Lambda$).

2. Missing data: we assume that the vertices divide into $c$ groups. We denote the group to which vertex $i$ belongs as $g_i$ and set of all missing data as $g$.

3. Model parameters: they describe the patterns of vertices in different groups ($\theta$, $\pi$).

\[ \theta_{ri} = \text{the probability that there exists an edge from a vertex in group } r \text{ to a vertex } i \]

\[ \pi_r = \text{the probability of a vertex belonging to group } r \]

\[ \sum_{r=1}^{c} \pi_r = 1 \quad \sum_{i=1}^{n} \theta_{ri} = 1 \]
A Likelihood Problem

The likelihood of the data given the model is

\[ Pr(A, g|\pi, \theta) = Pr(A|g, \pi, \theta) Pr(g|\pi, \theta), \]

where

\[ Pr(A|g, \pi, \theta) = \prod_{ij} \theta_{g_j,i}^{A_{ij}} \text{ and } Pr(g|\pi, \theta) = \prod_j \pi_{g_j}. \]

Frequently, one works not with the likelihood itself, but with the log-likelihood.

\[ \mathcal{L} = \ln Pr(A, g|\pi, \theta) = \sum_j \left[ \ln \pi_{g_j} + \prod_i \theta_{g_j,i}^{A_{ij}} \right]. \]
We cannot directly observe $g$.

It is, however, possible to calculate an expected value for the log-likelihood over all possible values of $g$.

\[
\overline{L} = \sum_{g_1=1}^{c} \cdots \sum_{g_n=1}^{c} \Pr(g|A, \theta, \pi) \sum_{j=1}^{n} \left[ \ln \pi_{g_j} + \sum_{i=1}^{n} A_{ij} \ln \theta_{g_j,i} \right]
\]

\[
\overline{L} = \sum_{r=1}^{c} \sum_{j=1}^{n} q_{rj} \left[ \ln \pi_{r} + \sum_{i=1}^{n} A_{ij} \ln \theta_{ri} \right]
\]

where

\[
q_{jr} = \Pr(g_{j} = r|A, \pi, \theta) = \frac{\Pr(A, g_{j} = r|\pi, \theta)}{\Pr(A|\pi, \theta)} = \frac{\pi_{r} \prod_{i} \theta_{ri}^{A_{ij}}}{\sum_{s} \pi_{s} \prod_{i} \theta_{si}^{A_{ij}}}
\]
The EM Algorithm

- Initialize model parameters (\( \theta, \pi \)) with random values.
- Find the probability a given vertex \( j \) is a member of group \( r \) (E-step).

\[
q_{jr} = \frac{\pi_r \prod_i \theta_{ri}^{A_{ij}}}{\sum_s \pi_s \prod_i \theta_{si}^{A_{ij}}}. 
\]

- Maximize the model parameters (M-step)

\[
\pi_r = \frac{1}{n} \sum_i q_{ir} \quad \theta_{ri} = \frac{\sum_j A_{ij} q_{jr}}{\sum_j k_j q_{jr}}.
\]

- Iterate until convergence.
The AddHealth Network
Example: Karate Club
Example: Word Network
Example: Assortative/Dissassortative Network

- **assortative**
- **disassortative**

Graph showing the relationship between success and the probability ratio $p_{out}/p_{in}$ for different values of $p$.

- **Key**:
  - Black dots: this paper
  - Red squares: max. modularity
  - Blue triangles: min. modularity

The graph illustrates the transition from assortative to disassortative networks as the probability ratio changes.
Example: AddHealth
Example: A “Keystone” Network
Example: A “Keystone” Network
“Big Ten” Results with the EM Algorithm

Vertex shading based on probability of being assigned to group 1.

Vertex size based on probability of being beaten by teams assigned to group 1.
Example: The “Big Ten” Conference

- Illinois
- Purdue
- Indiana
- Wisconsin
- Iowa
- Pennsylvania State
- Michigan State
- Northwestern
- Ohio State
- Michigan
- Minnesota