Synchronization and Pattern Evolution on Networks: The Interplay of Structure and Dynamics

Ben Johnson

Graduate Group in Applied Math

April 28, 2009
Dynamics on Networks

What does this mean?

- Focus on processes (diffusion, synchronization, proliferation) occurring on networks
- Functionality and efficiency of such processes relative to network topology and dynamics

Why do we care?

- Understand real-world networks
- Find a connection between structure and function
What does this mean?

- Focus on processes (diffusion, synchronization, proliferation) occurring on networks
- Functionality and efficiency of such processes relative to network topology and dynamics

Why do we care?

- Understand real-world networks
- Find a connection between structure and function
Introduction

Synchronization

Pattern Evolution

Dynamics on Networks

What does this mean?

- Focus on processes (diffusion, synchronization, proliferation) occurring on networks
- Functionality and efficiency of such processes relative to network topology and dynamics

Why do we care?

- Understand real-world networks
- Find a connection between structure and function
Dynamics on Networks

What does this mean?

- Focus on processes (diffusion, synchronization, proliferation) occurring on networks
- Functionality and efficiency of such processes relative to network topology and dynamics

Why do we care?

- Understand real-world networks
- Find a connection between structure and function
Dynamics on Networks

What does this mean?

- Focus on processes (diffusion, synchronization, proliferation) occurring on networks
- Functionality and efficiency of such processes relative to network topology and dynamics

Why do we care?

- Understand real-world networks
- Find a connection between structure and function
Dynamics on Networks

What does this mean?

• Focus on processes (diffusion, synchronization, proliferation) occurring on networks
• Functionality and efficiency of such processes relative to network topology and dynamics

Why do we care?

• Understand real-world networks
• Find a connection between structure and function
**Problem:** "Dynamics on Networks" is very broad

**Solution:** Narrow our focus

- Synchronization
- Pattern evolution

What is their relation to network connectivity and topology?
Problem: "Dynamics on Networks" is very broad

Solution: Narrow our focus

• Synchronization
• Pattern evolution

What is their relation to network connectivity and topology?
Synchronization

What is synchronization?
- Intuitive answer: Highly similar behavior

Can we be more precise?
- Exact
- Generalized synchronization
- Phase
- Lag
- Anticipatory
What is synchronization?

- Intuitive answer: Highly similar behavior

Can we be more precise?

- Exact
- Generalized synchronization
- Phase
- Lag
- Anticipatory
What is synchronization?

- Intuitive answer: Highly similar behavior

Can we be more precise?

- Exact
- Generalized synchronization
- Phase
- Lag
- Anticipatory
What is synchronization?
- Intuitive answer: Highly similar behavior

Can we be more precise?
- Exact
  - Generalized synchronization
- Phase
- Lag
- Anticipatory
Synchronization

What is synchronization?
- Intuitive answer: Highly similar behavior

Can we be more precise?
- Exact
- Generalized synchronization
  - Phase
  - Lag
  - Anticipatory
What is synchronization?

- Intuitive answer: Highly similar behavior

Can we be more precise?

- Exact
- Generalized synchronization
- Phase
  - Lag
- Anticipatory
Synchronization

What is synchronization?
- Intuitive answer: Highly similar behavior

Can we be more precise?
- Exact
- Generalized synchronization
- Phase
- Lag
- Anticipatory
What is synchronization?
- Intuitive answer: Highly similar behavior

Can we be more precise?
- Exact
- Generalized synchronization
- Phase
- Lag
- Anticipatory
Importance of Synchronization

- Synchronization is observed in many real-world networks
  - Fireflies flashing together
  - Neurons firing in a neural network
  - Heart pacemaker cells
  - Coupled laser arrays
- Understanding these may shed light on other networks
  - Connection to network structure?
  - Reveal unseen function
Synchronization itself is very broad, simplify analysis by using the Kuramoto model:

- Established, standard model for synchronization
- Well studied
- Simple, yet robust
Kuramoto Model

Given $N$ coupled oscillators, whose dynamics satisfy

$$\frac{d\phi_i}{dt} = \omega_i + \sum_{j=1}^{N} J_{ij} \sin(\phi_j - \phi_i) + I_{i,m}$$

- $\phi_i(t)$ = phase of oscillator $i$ at time $t$
- $\omega_i$ = natural frequency of oscillator $i$
- $J_{ij}$ = coupling strength between oscillators $i$ and $j$
- $I_{i,m}$ = external driving strength to oscillator $i$ for driving condition $m$
Perturbations Near Synchronization

Consider the difference between the perturbed and unperturbed system

\[
D_{i,m} = \Omega_m - \Omega_0 - I_{i,m} \\
= \sum_{j=1}^{N} J_{ij} [\sin(\phi_{j,m} - \phi_{i,m}) - \sin(\phi_{j,0} - \phi_{i,0})] \\
\approx \sum_{j=1}^{N} L_{ij} \theta_{j,m}
\]

- \( L \) is the Laplacian matrix
- \( \Omega_m \) and \( \Omega_0 \) are the driven and undriven collective frequencies
- \( \theta_{j,m} = \phi_{j,m} - \phi_{j,0} \)
Results

- Each driving condition $m$ yields $N - 1$ independent phase shifts $(\theta_{i,m})$ and a collective frequency $\Omega_m$
- Gives $N$ of possible $N^2$ network connections
- $M$ driving conditions provide $MN$ restrictions $\Rightarrow$ need at most $N$ experimental runs
- Reveals strength of connection
Results

• Each driving condition $m$ yields $N - 1$ independent phase shifts $(\theta_{i,m})$ and a collective frequency $\Omega_m$
• Gives $N$ of possible $N^2$ network connections
• $M$ driving conditions provide $MN$ restrictions $\Rightarrow$ need at most $N$ experimental runs
• Reveals strength of connection
Results

- Each driving condition $m$ yields $N - 1$ independent phase shifts $(\theta_{i,m})$ and a collective frequency $\Omega_m$
- Gives $N$ of possible $N^2$ network connections
- $M$ driving conditions provide $MN$ restrictions $\Rightarrow$ need at most $N$ experimental runs
- Reveals strength of connection
Results

- Each driving condition $m$ yields $N - 1$ independent phase shifts $(\theta_{i,m})$ and a collective frequency $\Omega_m$
- Gives $N$ of possible $N^2$ network connections
- $M$ driving conditions provide $MN$ restrictions $\Rightarrow$ need at most $N$ experimental runs
- Reveals strength of connection
Difficulties

- Difficult to solve $D = L \theta$ (ill-conditioned)
- Network size
- Cost of each experiment

How can we improve this method?
• Realize that most networks do not have $N^2$ connections
• Use singular value decomposition to create the matrix $\hat{J}$ and minimize $\|\hat{J}\|_1$
• Result: sparsest matrix that satisfies the system equations (minimal connections)
Improvement

- Realize that most networks do not have $N^2$ connections
- Use singular value decomposition to create the matrix $\hat{J}$ and minimize $\|\hat{J}\|_1$
- Result: sparsest matrix that satisfies the system equations (minimal connections)
Improvement

- Realize that most networks do not have $N^2$ connections.
- Use singular value decomposition to create the matrix $\hat{J}$ and minimize $\|\hat{J}\|_1$.
- Result: sparsest matrix that satisfies the system equations (minimal connections).
Quality of reconstruction

Element-wise difference between real and computed connectivity matrices:

$$\Delta J_{ij} : \frac{|J_{ij}^{\text{derived}} - J_{ij}^{\text{actual}}|}{2J_{\text{max}}}$$

Quality of reconstruction to accuracy $\alpha$ after $M$ experiments:

$$Q_\alpha(M) := \frac{1}{N^2} \sum_{i,j} H((1 - \alpha) - \Delta J_{ij}),$$

where $H$ is the Heaviside step function ($H(x) = 1$ for $x \geq 0$).
Figure: Quality of reconstruction and required number of experiments. Quality of reconstruction (α = .95) for k = 10 and N = 24(◇), N = 36(△), N = 66(○), and N = 96(○)
Minimum Number of Experiments

Minimum number of experiments for accurate reconstruction on quality level $q$:

$$M_{q,\alpha} := \min\{M : Q_\alpha(M) \geq q\}$$

- Assuming $0 < 1 - \alpha \ll 1$ and $0 < 1 - q \ll 1$
- Sublinear in numerical experiments
- Connectivity can be determined even if $M \ll N$
Minimum Number of Experiments

\[ M_{0.90, 0.95} \]

**Figure:** Minimum number of experiments required \((q = 0.90, \alpha = 0.95)\) versus network size \(N\) with best linear and logarithmic fits (gray and black solid lines). Inset show same data with \(N\) on logarithmic scale.
Community detection

Synchronization dynamics can reveal the connectivity of a network.

Very often, we wish to know more than just connectivity. Can we detect community structure as well?
Community detection

Synchronization dynamics can reveal the connectivity of a network.

Very often, we wish to know more than just connectivity. Can we detect community structure as well?
Start with Kuramoto model for coupled oscillators:

\[
\frac{d\phi_i}{dt} = \omega_i + \sum_{j=1}^{N} J_{ij} \sin(\phi_j - \phi_i) + I_{i,m}
\]

With $I_{i,m} = 0$ (undriven network)

Look at average correlation between pairs of nodes. Define local order parameter:

\[
\rho_{ij}(t) = \langle \cos(\phi_i(t) - \phi_j(t)) \rangle
\]

Why cosine?
Start with Kuramoto model for coupled oscillators:

\[ \frac{d\phi_i}{dt} = \omega_i + \sum_{j=1}^{N} J_{ij} \sin(\phi_j - \phi_i) + I_{i,m} \]

With \( I_{i,m} = 0 \) (undriven network)

Look at average correlation between pairs of nodes. Define local order parameter:

\[ \rho_{ij}(t) = \langle \cos(\phi_i(t) - \phi_j(t)) \rangle \]

Why cosine?
• Start with Kuramoto model for coupled oscillators:

\[
\frac{d\phi_i}{dt} = \omega_i + \sum_{j=1}^{N} J_{ij} \sin(\phi_j - \phi_i) + I_{i,m}
\]

With \( I_{i,m} = 0 \) (undriven network)

• Look at average correlation between pairs of nodes. Define local order parameter:

\[
\rho_{ij}(t) = \langle \cos(\phi_i(t) - \phi_j(t)) \rangle
\]

• Why cosine?
Dynamic Connectivity Matrix

Convert correlation matrix $[\rho_{ij}]$ into a binary matrix.

Define

$$D_t(T)_{ij} = \begin{cases} 1 & \text{if } \rho_{ij}(t) \geq T \\ 0 & \text{if } \rho_{ij}(t) < T \end{cases}$$

$T$ is some threshold value.

- Different values of $T$ reveal different levels of structure in the network
- Fix a threshold $T$ and look at time evolution
Convert correlation matrix $[\rho_{ij}]$ into a binary matrix.

Define

$$D_t(T)_{ij} = \begin{cases} 1 & \text{if } \rho_{ij}(t) \geq T \\ 0 & \text{if } \rho_{ij}(t) < T \end{cases}$$

$T$ is some threshold value.

- Different values of $T$ reveal different levels of structure in the network
- Fix a threshold $T$ and look at time evolution
Visualization of Dynamic Connectivity

What are the communities of this network?

Red for shorter times, blue for longer times
Visualization of Dynamic Connectivity

What are the communities of this network?

Red for shorter times, blue for longer times
What about this network?
What about this network?

Fig. 1. Network with an inhomogeneous distribution of communities. a) the network structure; b) eigenvalues spectra and number of detected communities as a function of time; c) dendrogram of the community merging. d) time needed for each pair of oscillators to synchronize. Red for shorter times, blue for larger times.

This type of networks, apart from its hierarchical structure has some nodes with a special role in terms of number of connections (hubs) in contrast to the networks discussed previously that are essentially homogeneous in degree. In Fig. 3a we present a very simple example of this class of networks for the case of two hierarchical levels.

In a previous work [21] we represented the correlation matrix of the system $\rho_{ij}(t)$ at the same time instant $t$ for two slightly different two level hierarchical networks with structure 13-4 and 15-2. From that representation, we could identify the two levels of the hierarchical distribution of communities.
Examples

And this network?
Examples

And this network?
Results

- Accurately detects the community structure of a network
- Also detects substructure within communities
- Reveals equivalence between disconnected communities
Pattern Evolution

Dynamics can reveal a lot of information about network connectivity and community structure.

Can network structure predict the behavior of the dynamics?
Pattern Evolution

Dynamics can reveal a lot of information about network connectivity and community structure.

Can network structure predict the behavior of the dynamics?
Scale-Free Networks

Recall that our degree distribution follows a power law:

$$P(k) \sim k^{-\gamma}$$

For our purposes (and in many real-world networks) $2 < \gamma < 3$
Our Model

- Undirected network with scale-free degree distribution
- Vertex degree governed by $k_0 \leq k \leq k_{\text{max}}$ with $k_0 \geq 2$ and $k_{\text{max}} \sim N^{1/\gamma-1}$
- Average vertex degree $\langle k \rangle \geq 10$
- Each vertex has a binary, Ising-like spin variable
Our Model

- Undirected network with scale-free degree distribution
- Vertex degree governed by $k_0 \leq k \leq k_{\text{max}}$ with $k_0 \geq 2$ and $k_{\text{max}} \sim N^{1/\gamma-1}$
- Average vertex degree $\langle k \rangle \geq 10$
- Each vertex has a binary, Ising-like spin variable
Our Model

- Undirected network with scale-free degree distribution
- Vertex degree governed by $k_0 \leq k \leq k_{\text{max}}$ with $k_0 \geq 2$ and $k_{\text{max}} \sim N^{1/\gamma-1}$
- Average vertex degree $\langle k \rangle \geq 10$
- Each vertex has a binary, Ising-like spin variable
Our Model

- Undirected network with scale-free degree distribution
- Vertex degree governed by $k_0 \leq k \leq k_{\text{max}}$ with $k_0 \geq 2$ and $k_{\text{max}} \sim N^{1/\gamma - 1}$
- Average vertex degree $\langle k \rangle \geq 10$
- Each vertex has a binary, Ising-like spin variable
Time evolution

We use local majority dynamics

- State of vertex \(i\) at time \(t\) is \(\sigma_i(t) = \pm 1\).
- Evolution of system:

\[
\sigma_i(t + 1) = \begin{cases} 
+1 & \text{if } h_i(t) > 0 \\
-1 & \text{if } h_i(t) < 0 \\
\pm 1 & \text{with } P = \frac{1}{2} \text{ if } h_i(t) = 0
\end{cases}
\]

- \(h_i(t) = \sum_{j \in J_i} \sigma_j(t)\) with \(J_i = \{\text{nodes connected to vertex } i\}\).
We use local majority dynamics

- State of vertex $i$ at time $t$ is $\sigma_i(t) = \pm 1$.
- Evolution of system:

$$\sigma_i(t+1) = \begin{cases} 
+1 & \text{if } h_i(t) > 0 \\
-1 & \text{if } h_i(t) < 0 \\
\pm 1 \text{ with } P = \frac{1}{2} & \text{if } h_i(t) = 0
\end{cases}$$

- $h_i(t) = \sum_{j \in J_i} \sigma_j(t)$ with $J_i = \{\text{nodes connected to vertex } i\}$. 
**Time evolution**

We use local majority dynamics

- State of vertex $i$ at time $t$ is $\sigma_i(t) = \pm 1$.
- Evolution of system:

  $$\sigma_i(t + 1) = \begin{cases} 
  +1 & \text{if } h_i(t) > 0 \\
  -1 & \text{if } h_i(t) < 0 \\
  \pm 1 \text{ with } P = \frac{1}{2} & \text{if } h_i(t) = 0
  \end{cases}$$

- $h_i(t) = \sum_{j \in J_i} \sigma_j(t)$ with $J_i = \{\text{nodes connected to vertex } i\}$. 

To study evolution patterns, consider

- $q_k(t) = \text{probability that a vertex of degree } k \text{ is } +1$
- $Q(t) = \text{probability that for any vertex chosen, a random neighbor is } +1$

A vertex associated with a random edge has degree $= k$ with probability $\frac{kp(k)}{\sum_k kp(k)} = \frac{kp(k)}{\langle k \rangle}$.

Then

$$Q(t) = \sum_k \frac{kp(k)}{\langle k \rangle} q_k(t)$$
To study evolution patterns, consider

- \( q_k(t) = \text{probability that a vertex of degree } k \text{ is } +1 \)
- \( Q(t) = \text{probability that for any vertex chosen, a random neighbor is } +1 \)

A vertex associated with a random edge has degree = \( k \) with probability \( \frac{kP(k)}{\langle k \rangle} = \frac{kP(k)}{\langle k \rangle} \).

Then

\[
Q(t) = \sum_k \frac{kP(k)}{\langle k \rangle} q_k(t)
\]
To study evolution patterns, consider

- $q_k(t) = \text{probability that a vertex of degree } k \text{ is } +1$
- $Q(t) = \text{probability that for any vertex chosen, a random neighbor is } +1$

A vertex associated with a random edge has degree $= k$ with probability $\frac{kP(k)}{\sum_k kP(k)} = \frac{kP(k)}{\langle k \rangle}$.

Then

$$Q(t) = \sum_k \frac{kP(k)}{\langle k \rangle} q_k(t)$$
To study evolution patterns, consider

- \( q_k(t) = \text{probability that a vertex of degree } k \text{ is } +1 \)
- \( Q(t) = \text{probability that for any vertex chosen, a random neighbor is } +1 \)

A vertex associated with a random edge has degree = \( k \) with probability

\[
\frac{kP(k)}{\sum_k kP(k)} = \frac{kP(k)}{\langle k \rangle}.
\]

Then

\[
Q(t) = \sum_k \frac{kP(k)}{\langle k \rangle} q_k(t)
\]
Given our previous description of local majority dynamics, we see

\[ q_k(t + 1) = \sum_{m = \lceil k/2 \rceil}^k \left[ 1 - \frac{1}{2} \delta_{m, k/2} \right] \binom{k}{m} Q^m(t) [1 - Q(t)]^{k-m} \]

and

\[ \psi(\mathcal{Q}) = Q(t + 1) = \sum_k \frac{kP(k)}{\langle k \rangle} q_k(t + 1) \]
It is easy to check that $Q$ has 3 fixed points: 0, $\frac{1}{2}$, and 1.

- 0 and 1 are both stable (all + or all - system)
- $\frac{1}{2}$ is unstable phase boundary between attracting fixed points

Define order parameter $y(t) = |Q(t) - \frac{1}{2}|$
Evolution of order parameter

Working with the equations of our model, we find that

$$y(t + 1) \approx \Psi' \left( \frac{1}{2} \right) y(t)$$

$$\Psi' \left( \frac{1}{2} \right) \approx \begin{cases} c_{\gamma} k_0^{1/2} & \text{for } \gamma > \frac{5}{2} \\ c_{\gamma} k_0^{1/2} \ln N & \text{for } \gamma = \frac{5}{2} \\ c_{\gamma} k_0^{1/2} N^{\alpha/2} & \text{for } 2 < \gamma < \frac{5}{2} \end{cases}$$

where \( \alpha = \frac{5 - 2\gamma}{\gamma - 1} \)
Analytical results

Starting with a strongly disordered state \( y(t_0) = \pm \frac{1}{N} \) evolve system using local majority rule dynamics.

Define \( t_d \) as the time to reach \( y^* \), that satisfies \( |y^*| \geq \frac{1}{4} \)

From analysis of our evolution equations, we find \( t_d \approx \frac{\ln(<k>N)}{\ln(\Psi'(1/2))} \)

\[
\begin{align*}
t_d \sim & \quad \ln N \quad \text{for } \gamma > \frac{5}{2} \\
& \frac{\ln N}{\ln(\ln N)} \quad \text{for } \gamma = \frac{5}{2} \\
& 2^{\frac{\gamma-1}{5-2\gamma}} \quad \text{for } 2 < \gamma < \frac{5}{2}
\end{align*}
\]
Figure: $\gamma = 2.25$ and $k_0 = 5$ (●), $\gamma = 3$ and $k_0 = 10$ (■), Poissonian network (♦). $N = 2^{18}$, $\langle k \rangle = 20$. 
**Figure:** $\gamma = 2.25$ and $k_0 = 5$ (○), $\gamma = 3$ and $k_0 = 10$ (■), Poissonian network (◆).
Figure: \( \gamma = 2.25 \) and \( k_0 = 5 \) (●), \( \gamma = 2.5 \) and \( k_0 = 7 \) (■), \( \gamma = 3 \) and \( k_0 = 10 \) (◇), \( \langle k \rangle = 20 \). Filled = numerical, empty = analytic
Results

- Numerical simulations agree with analysis of evolution equations
- We don’t find domains with different patterns (no meta-stability)
- In all numerical runs, the probability of not reaching a completely ordered pattern is less than $10^{-2}$
- Decrease in mean vertex degree ($\langle k \rangle$) increases decay time
Changing existing patterns

Given a network in an all-spin-down pattern, how many flips to cause evolution into all-spin-up pattern?

- Simple-minded approach: Choose random vertices - Requires $\sim N/2$ flips
- Better approach: Choose mostly highly connected vertices

Analytic results:

$$\Omega_{\text{min}} \approx 2^{-\left(\gamma-1\right)/\left(\gamma-2\right)}$$

Note that

$$\lim_{\gamma \to 2^+} \Omega_{\text{min}} = 0 \quad \text{and} \quad \lim_{\gamma \to \infty} \Omega_{\text{min}} = \frac{1}{2}$$
Figure: Minimal fraction $\Omega_{\text{min}}$ of spins that must be flipped to induce transition from all-spin-down to all-spin-up pattern. $N = 10^5$. Open squares = analytic results, Filled squares = numerical results.
Big picture

- \( \gamma = \frac{5}{2} \) represents a sharp boundary for pattern evolution on scale-free networks.
- For \( 2 < \gamma < \frac{5}{2} \) strongly disordered patterns decay in finite even in the limit of large \( N \)
- Not the case for \( \gamma \geq \frac{5}{2} \)

Many real-world networks have \( 2 < \gamma < \frac{5}{2} \). Why?
Big picture

- $\gamma = \frac{5}{2}$ represents a sharp boundary for pattern evolution on scale-free networks.
- For $2 < \gamma < \frac{5}{2}$ strongly disordered patterns decay in finite even in the limit of large $N$
- Not the case for $\gamma \geq \frac{5}{2}$

Many real-world networks have $2 < \gamma < \frac{5}{2}$. Why?
Big picture

- $\gamma = \frac{5}{2}$ represents a sharp boundary for pattern evolution on scale-free networks.
- For $2 < \gamma < \frac{5}{2}$ strongly disordered patterns decay in finite even in the limit of large $N$.
- Not the case for $\gamma \geq \frac{5}{2}$.

Many real-world networks have $2 < \gamma < \frac{5}{2}$. Why?
Big picture

- $\gamma = \frac{5}{2}$ represents a sharp boundary for pattern evolution on scale-free networks.
- For $2 < \gamma < \frac{5}{2}$ strongly disordered patterns decay in finite even in the limit of large $N$.
- Not the case for $\gamma \geq \frac{5}{2}$.

Many real-world networks have $2 < \gamma < \frac{5}{2}$. Why?
Where to go from here

- Weighted edges in network
- Effect of clustering and modularity
- Dynamic topology
- Interaction delays
- Multi-layered network