A Review of Turbulence Modeling and Nonextensive Statistical Mechanics

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In the 1988, Dr. Tsallis first proposed a new approach to the development of entropy in a system called nonextensive statistical mechanics. The basic philosophy behind this method is that the system and its geometrical and dynamical properties are needed in order to determine entropy and the system's thermodynamic characteristics [5]. It is not an alternative to the well-known Boltzmann-Gibbs equation, but a generalization of the function that can be especially applicable for more complex systems such as turbulence modeling. There have been many experiments proving the validity of the theory, and with each experiment, the model becomes more accurate [1]. Given time, nonextensive statistical mechanics may be a useful tool for developing more accurate and efficient algorithms for turbulent simulations.

The Boltzmann-Gibbs equation (1) is widely used in physics to determine the various statistical properties of a system by maximizing the Boltzmann entropy. The solution is solely dependent on the amount of microstates in a given macrostate, W, the probability of each microstate, and Boltzmann's constant. The formula was derived from the idea that entropy is always additive as shown in equation (2) [5].

$$S_{BG} = -k \sum_{i=1}^{W} p_i \ln p_i \tag{1}$$

$$S_{BG}(A+B) = S_{BG}(A) + S_{BG}(B) \tag{2}$$

Contrary to this view, nonextensive statistical mechanics models the entropy as non-additive by including a nonextensive parameter, q, which allows the entropy to be flexible or dependent on the type of system analyzed. The more generalized entropy is shown in equation (3). If macrostates A and B are independent, then they can still be combined by the following method in equation (4) [5].

$$S_q = k\frac{1-\sum_{i=1}^{W} p_i^q}{q-1} \tag{3}$$

$$S_q(A+B) = S_q(A) + S_q(B) + (1-q)\frac{S_q(A)S_q(B)}{k} \tag{4}$$

Note that if q = 1, the Boltzmann-Gibbs equation is recovered and the entropy is additive. For complex systems, q ≠ 1 and there is some degree of variation in the thermodynamic properties related by a power law. Examples of complex systems include systems with long-ranged interactions/correlations, multifractality, metastability, and non-equilibrium properties [1]. For these systems, q must be determined experimentally to provide the information for further studies in a given system [5].

It is thought that turbulent systems have reoccurring fractal patterns in the wake of large eddies [4]. This makes nonextensive statistical mechanics important for describing the properties of turbulence. Dr. Daniels, Beck and Bodenschatz analyze the use of such methods in describing defect turbulence. A thin layer of fluid is inclined between 15° and 75° and heated and cooled from above and below, respectively to

simulate convection. This is shown in Figure 1 [1].

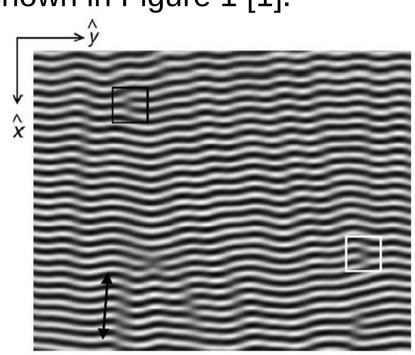

Fig. 1. Example shadowgraph image of undulation chaos in fluid layer heated from below and cooled from above, inclined by an angle $\theta = 30°$. The nondimensional driving parameter is $\epsilon = 0.08$. Black box encloses a positive defect; white, a negative. Arrow is adjacent to a tearing region of low-amplitude convection. Uphill direction is at left side of page. Region shown is the subregion of size $51d \times 63d$ used for analysis.

Locations of defect velocity begin to form and are measured. Given a nonextensive parameter of $q \approx 1.5$, it was shown that the probability distributions of defect velocities match well with the distributions formulated using Tsallis statistics [1].

Another example of the application of nonextensive statistical mechanics to turbulence is Dr. Jung, Storey, Aubert, and Swinney's work on rotating quasi, two-dimensional turbulent flow. The large vortices created by experiments with fluid flow in a rotating annulus led to the idea of Tsallis statistics being applicable to systems with long-range interactions. Assuming conservation of energy and enstrophy, both the extensive and nonextensive models were compared with the experimental results for two-dimensional inviscid flow. When analyzing the vorticity probability density function, the nonextensive model produced accurate correlations while the extensive model did not. The nonextensive parameter obtained was $q = 1.9 \pm 0.2$. When an inner ring of sources and outer ring of sinks were simulated to create a strong turbulent circulation, the corresponding data yielded $q = 1.32 \pm 0.03$. Lastly, when a turbulent Couette-Taylor flow was analyzed with numerous spatial measurement points, comparing the velocity distribution resulted in $q = 1.17$. When the distance between points was increased, q decreased to unity [JUNG].

These experiments could potentially serve a great purpose for engineering applications such as the development of analysis tools for turbulent flow. Computational turbulent simulations use numerical methods to solve classic fluid models based on partial differential equations with initial and boundary conditions. Currently, turbulence is simulated using a multitude of different methods including Reynolds averaged Navier Stokes (RANS), algebraic correlations, large eddy simulation (LES), detached eddy simulation (DES), direct numerical simulation (DNS), or a combination of each [3]. Each method has its own advantages and disadvantages but they would all benefit from more accurate initial/boundary conditions. Given a certain application or system, an initial/boundary condition could be generated from a velocity, pressure, or vorticity distribution from nonextensive statistical mechanics. The simulation could take less time to converge to a final answer depending on the accuracy of the nonextensive parameter obtained from experiment. Finally, an analysis toolbox could be created to simulate a multitude of different turbulent models based on the various research experiments in turbulence.

Nonextensive statistical mechanics is quickly developing into the standard model for maximizing entropy and obtaining thermodynamic characteristics of complex systems. The results obtained from each experiment not only show that turbulence can be successfully described using methods from nonextensive statistical mechanics, but also that results can be used for future study in turbulent flow. It is believed that other defect turbulent systems exhibit similar behavior to the experiments conducted by Dr. Daniels and further investigations are needed to determine the validity of the model [1]. The nonextensive parameters found in the rotating annulus experiments could also be used for further study in the field [2]. Once a collection of experiments have been formulated, a proper turbulence toolbox could be created based on Tsallis statistics. This would increase the efficiency of turbulence modeling for engineering applications.

References

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