UC Davis, ECS20, Winter 2017

Discrete Mathematics for Computer Science

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(slides adopted from Michael Frank and Haluk Bingöl)

Lecture 8

Functions and Sequences
Rosen, Secs. 2.3, 2.4

Function: Formal Definition

- Def. For any sets A, B, we say that a *function f (or "mapping") from A to B* is a particular assignment of exactly one element $f(x) \in B$ to each element $x \in A$. We can write

$$f: A \rightarrow B$$

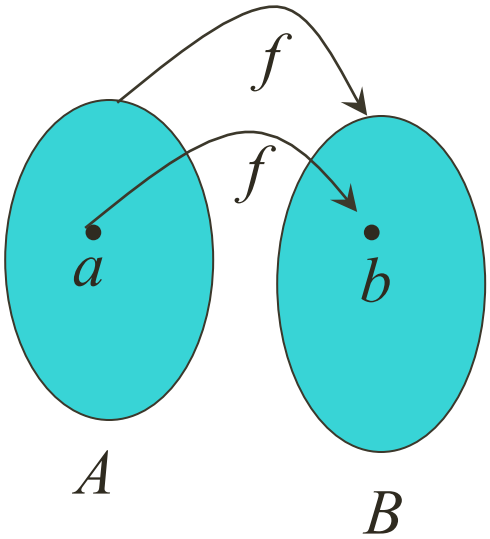
- For a $\in A$ and b $\in B$, we can evaluate $f(a)=b$.

Important Function Terminology

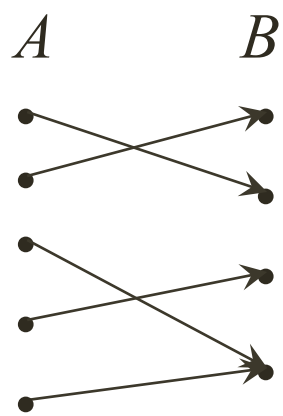
- Def. Let $f{:}A \rightarrow B$, and $f(a)=b$ (where $a \in A$ & $b \in B$). Then
  - A is the *domain* of f.
  - B is the *codomain* of f.
  - b is the *image* of a under f.
  - a is a *pre-image* of b under f.
    - In general, b may have more than 1 pre-image.
  - The *range* $R \subseteq B$ of f is $R=\{b \mid \exists a\ f(a)=b\}$.

Graphical Representations

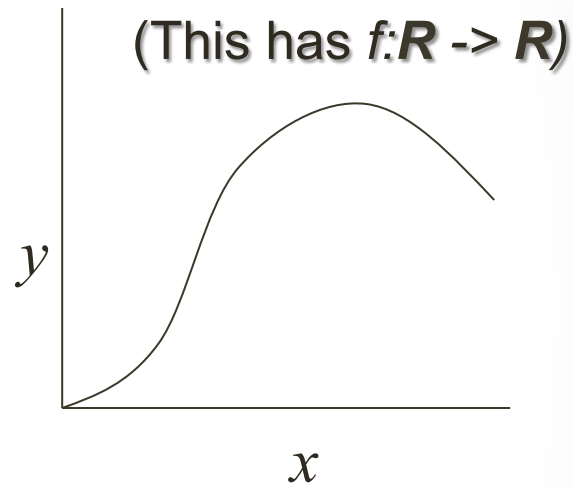
- Functions can be represented graphically in several ways:



Like Venn diagrams



Bipartite Graph

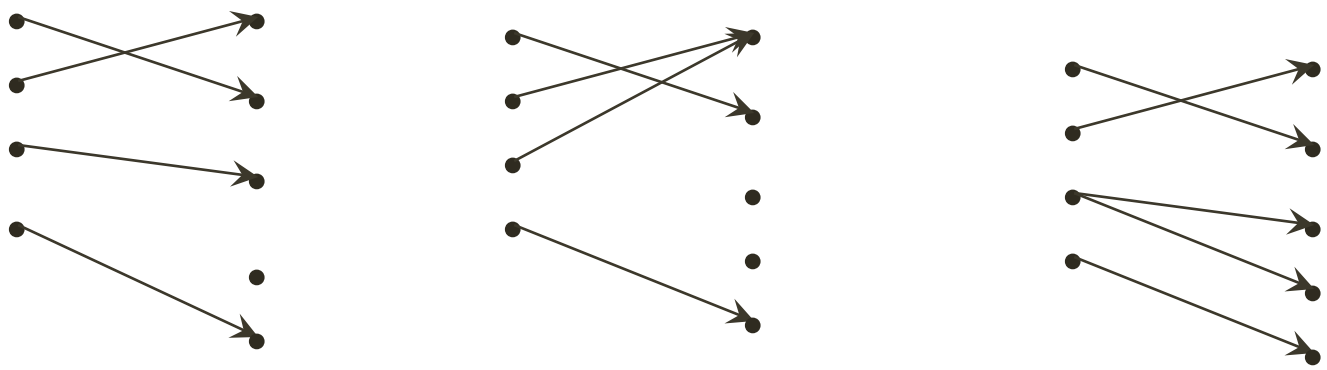


Plot

Note: EVERY element of set A has to be mapped to ONE (and only one) element in B.

One-to-One or "Injective" functions

- Bipartite (2-part) graph representations of functions that are (or are not) one-to-one:



One-to-one

Not one-to-one

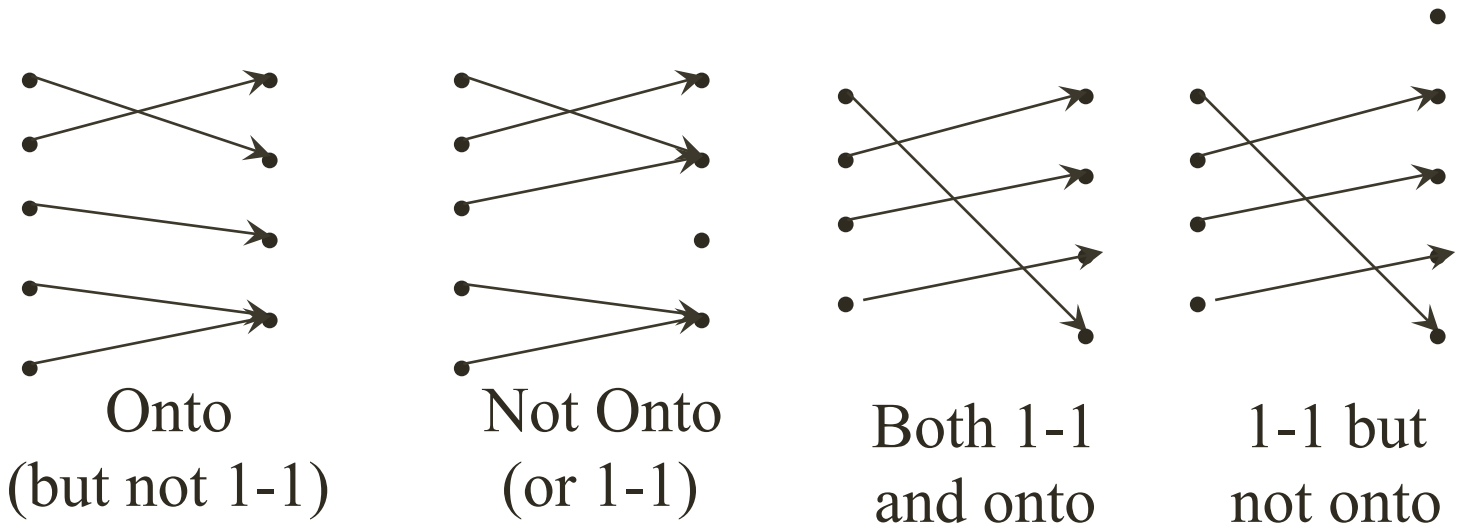
("Many-to-one" instead)

Not a function!

Note |domain| ≤ |codomain| for 1-to-1

Onto or "surjective" functions

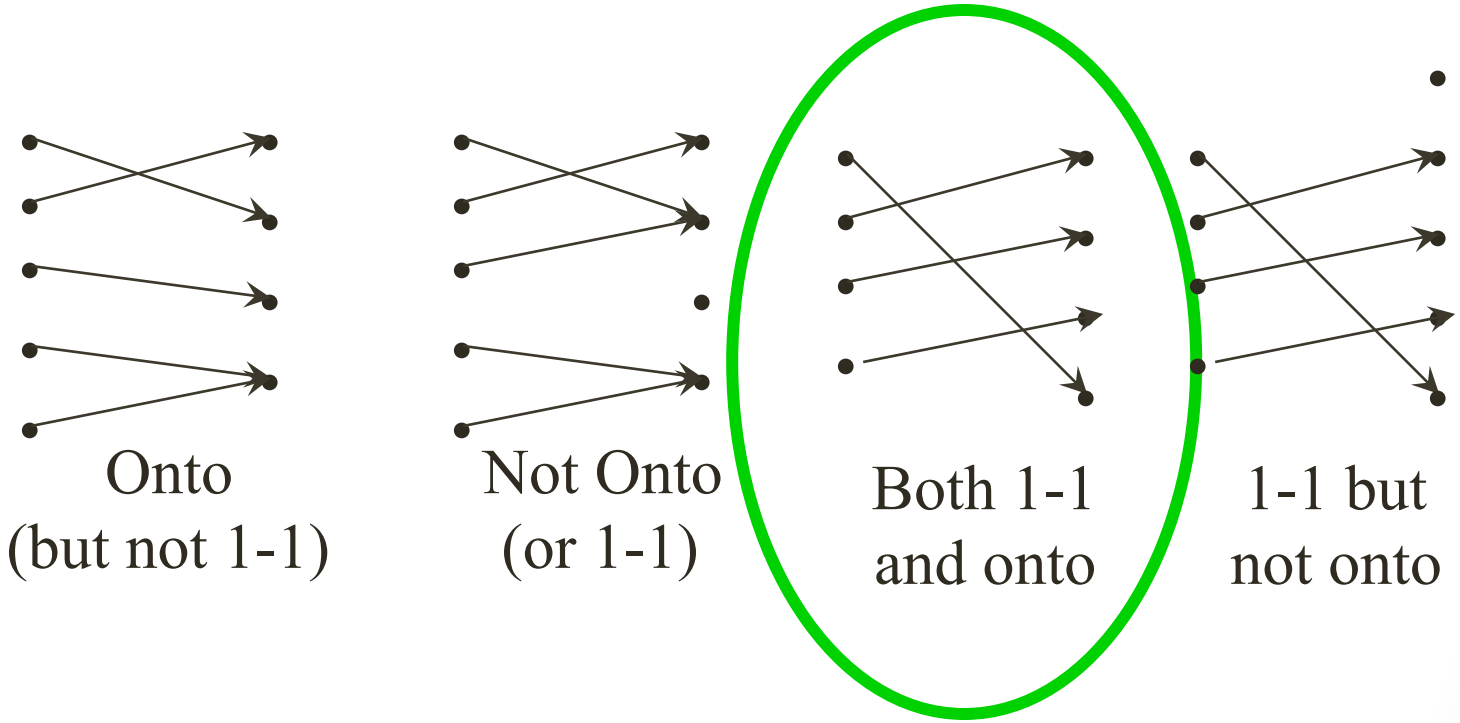
- Some functions that are, or are not, *onto* their codomains:



Note |domain| ≥ |codomain| for onto

1-1 and onto = bijection/invertible

- Some functions that are, or are not, *onto* their codomains:

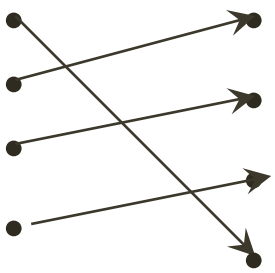


Bijections

- Def. A function f is said to be *a bijection*, (or *a one-to-one correspondence*, or *reversible*, or *invertible*,) iff it is both one-to-one and onto.

- Def. For bijections $f{:}A \to B$, there exists an *inverse of f*, written $f^{-1}{:}B \to A$, which is the unique function such that
  - (where I_A is the identity function on A)

$$f^{-1} \circ f = I_A$$



Both 1-1 and onto

|Domain| = |Codomain| = |Range|

We can invert the function!!

Graphs of Functions

- We can represent a function $f{:}A \rightarrow B$ as a set of ordered pairs $\{(a,f(a)) \mid a \in A\}$.

- Note that $\forall a$, there is only 1 pair (a,b).
  - Later (ch.6): *relations* loosen this restriction.

- For functions over numbers, we can represent an ordered pair (x,y) as a point on a plane.
  - A function is then drawn as a curve (set of points), with only one y for each x.

A Couple of Key Functions

- In discrete math, we will frequently use the following two functions over real numbers:

- Def. The *floor* function $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$, where $\lfloor x \rfloor$ ("floor of x") means the largest (most positive) integer $\leq x$. *Formally,* $\lfloor x \rfloor :\equiv \max(\{ j \in \mathbf{Z} \mid j \leq x\})$.

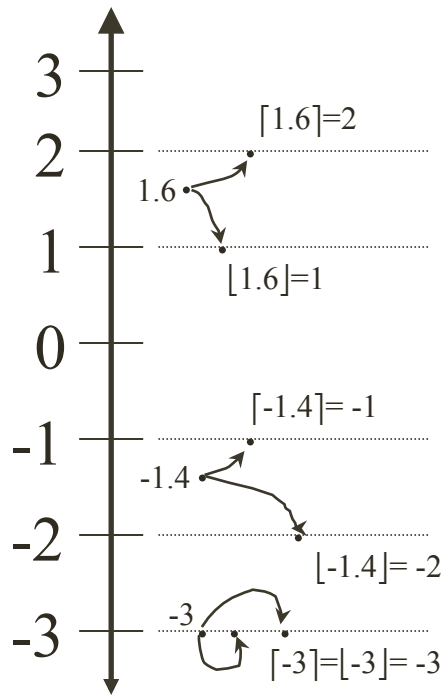
- Def. The *ceiling* function $\lceil \cdot \rceil : \mathbb{R} \to \mathbb{Z}$, where $\lceil x \rceil$ ("ceiling of x") means the smallest (most negative) integer $\geq x$. *Formally,* $\lfloor x \rfloor :\equiv \min(\{ j \in \mathbf{Z} \mid j \geq x\})$

Visualizing Floor & Ceiling

- Real numbers "fall to their floor" or "rise to their ceiling."

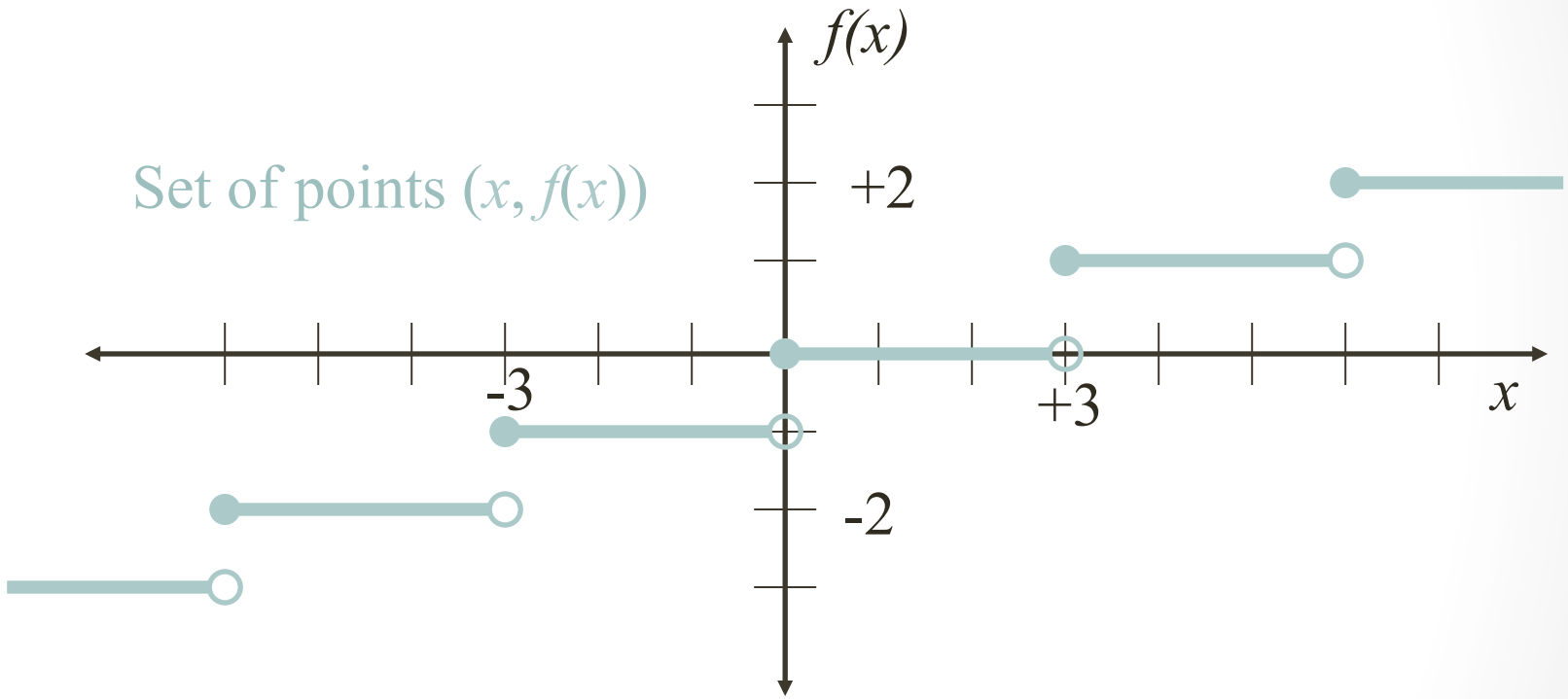
- Note that if $x \notin \mathbf{Z}$,
  $\lfloor -x \rfloor \neq -\lfloor x \rfloor$ &
  $\lceil -x \rceil \neq -\lceil x \rceil$

- Note that if $x \in \mathbf{Z}$,
  $\lfloor x \rfloor = \lceil x \rceil = x$.



Plots with floor/ceiling: Example

- Plot of graph of function $f(x) = \lfloor x/3 \rfloor$:



Note, the open dot denotes that in the limit the point is not on the curve, and the closed dot means it is on the curve.

Operators (general definition)

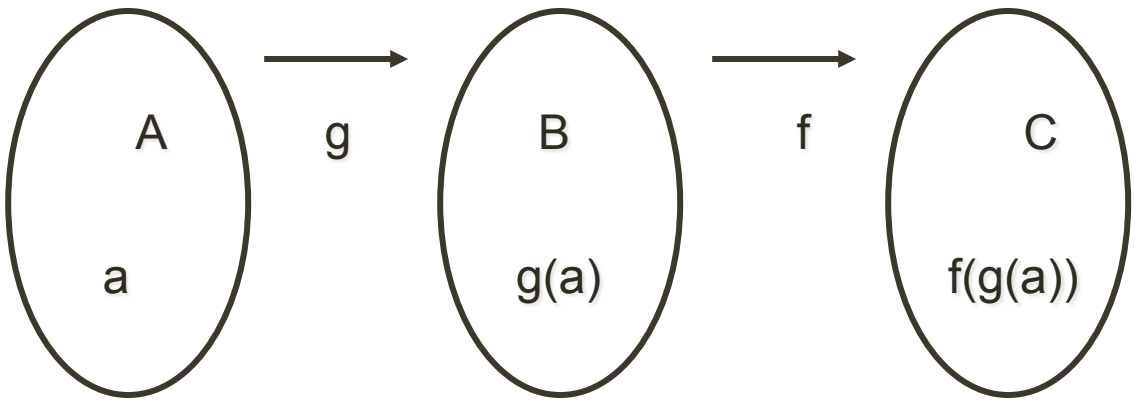
- Def. An *n*-ary *operator over* (or *on*) the set *S* is any function from the set of ordered *n*-tuples of elements of *S*, to *S* itself.
- Ex. If $S=\{\mathbf{T},\mathbf{F}\}$,
  $\neg$ can be seen as a unary operator, and
  $\wedge,\vee$ are binary operators on *S*.
- Ex. $\cup$ and $\cap$ are binary operators on the set of all sets. (See HW3 with $\cup_i^k$ notation)

Combining Function Operators

- $+, \times$ ("plus", "times") are binary operators over $\mathbb{R}$. (Normal addition & multiplication.)
- Therefore, we can also add and multiply *functions*
- Def. Let $f, g: \mathbb{R} \to \mathbb{R}$.
  - $(f + g): \mathbb{R} \to \mathbb{R}$, where $(f + g)(x) = f(x) + g(x)$
  - $(f \times g): \mathbb{R} \to \mathbb{R}$, where $(f \times g)(x) = f(x) \times g(x)$

Function Composition Operator

- Def. Let $g{:}A \rightarrow B$ and $f{:}B \rightarrow C$. *The composition of f and g*, denoted by $f \circ g$, is defined by $(f \circ g) = f(g(a))$.



- Remark. $\circ$ (like Cartesian $\times$, but unlike $+, \wedge, \cup$) is non-commuting. (Generally, $f \circ g \neq g \circ f$.)

Review of §2.3 (Functions)

- Function variables f, g, h, …
- Notations: $f{:}A \rightarrow B$, $f(a)$, $f(A)$.
- Terms: image, preimage, domain, codomain, range, one-to-one, onto, strictly (in/de)creasing, bijective, inverse, composition.
- Function unary operator f^{-1}, binary operators +, -, *etc.*, and $\circ$.
- The $\mathbf{R} \rightarrow \mathbf{Z}$ functions $\lfloor x \rfloor$ and $\lceil x \rceil$.

§2.4: Sequences, Strings, & Summations

- A *sequence* or *series* is just like an ordered *n*-tuple, except:
  - Each element in the series has an associated *index* number. *E.g.,* $(a_1, a_2, \ldots, a_k)$
  - A sequence or series may be infinite.
- A *string* is a sequence of *symbols* from some finite *alphabet*. (e.g., words in a language)
- A *summation* is a compact notation for the sum of all terms in a (possibly infinite) series.

$$\sum_{i=j}^{k} a_i :\equiv a_j + a_{j+1} + \ldots + a_k$$

Sequences

- Def. A *sequence* or *series* $\{a_n\}$ is identified with a *generating function* $f{:}S \rightarrow A$ for some subset $S \subseteq \mathbb{N}$ and for some set A.
  - Often we have $S = \mathbb{N}$ or $S = \mathbb{Z}^+ = \mathbb{N} - \{0\}$.

Recognizing Sequences

- Sometimes, you're given the first few terms of a sequence and you need to find the sequence's generating function, which is a procedure to enumerate the sequence.

- Examples: What's the next number?
- 1,2,3,4,…
- 1,3,5,7,9,…
- 2,3,5,7,11,…

5 (the 5th smallest number >0)

11 (the 6th smallest odd number >0)

13 (the 6th smallest prime number)

Sequence elements, a_n

Def. If f is a generating function for a series $\{a_n\}$, then for $n \in S$, the symbol a_n denotes $f(n)$, also called *term n* of the sequence.
The *index* of a_n is n. (Or, often i is used.)

A series is sometimes denoted by listing its first and/or last few elements, and using ellipsis (…) notation.
E.g., "$\{a_n\} = 0, 1, 4, 9, 16, 25, \ldots$" is taken to mean
$\forall n \in \mathbb{N}, a_n = n^2$.

Sequence Examples

- Some authors write "the sequence $a_1, a_2, \ldots$" instead of $\{a_n\}$, to ensure that the set of indices is clear. Be careful: often notation leaves the indices ambiguous, but context makes it clear.

- Ex. An example of an infinite series:
  - Consider the series $\{a_n\} = a_1, a_2, \ldots$, where $(\forall n \geq 1)\ a_n = f(n) = 1/n$.
  - Then, we have $\{a_n\} = 1, 1/2, 1/3, \ldots$

Switch to blackboard mode