Functions and Sequences
Rosen, Secs. 2.3, 2.4
Function: Formal Definition

• Def. For any sets A, B, we say that a function f (or “mapping”) from A to B is a particular assignment of exactly one element $f(x) \in B$ to each element $x \in A$. We can write

$$f: A \rightarrow B$$

• For $a \in A$ and $b \in B$, we can evaluate $f(a) = b$. 
Important Function Terminology

- **Def.** Let $f: A \rightarrow B$, and $f(a) = b$ (where $a \in A$ & $b \in B$). Then
  - $A$ is the *domain* of $f$.
  - $B$ is the *codomain* of $f$.
  - $b$ is the *image* of $a$ under $f$.
  - $a$ is a *pre-image* of $b$ under $f$.
  - In general, $b$ may have more than 1 pre-image.
  - The *range* $R \subseteq B$ of $f$ is $R = \{b \mid \exists a \ f(a) = b \}$. 
Graphical Representations

- Functions can be represented graphically in several ways:

Like Venn diagrams

Bipartite Graph

(This has \( f: \mathbb{R} \to \mathbb{R} \))

Note: EVERY element of set A has to be mapped to ONE (and only one) element in B.
One-to-One or “Injective” functions

- Bipartite (2-part) graph representations of functions that are (or are not) one-to-one:

  ![Graphs]

  One-to-one
  Not one-to-one
  Not a function!

  (“Many-to-one” instead)

Note $|\text{domain}| \leq |\text{codomain}|$ for 1-to-1
Onto or “surjective” functions

- Some functions that are, or are not, onto their codomains:

  - Onto (but not 1-1)
  - Not Onto (or 1-1)
  - Both 1-1 and onto
  - 1-1 but not onto

Note |domain| \(\geq|\text{codomain}|\) for onto
1-1 and onto = \textbf{bijection/invertible}

• Some functions that are, or are not, \textit{onto} their codomains:

- Onto (but not 1-1)
- Not Onto (or 1-1)
- Both 1-1 and onto
- 1-1 but not onto
Bijections

• **Def.** A function $f$ is said to be a *bijection*, (or a *one-to-one correspondence*, or *reversible*, or *invertible*,) iff it is both one-to-one and onto.

• **Def.** For bijections $f:A \rightarrow B$, there exists an *inverse of $f$*, written $f^{-1}:B \rightarrow A$, which is the unique function such that
  * (where $I_A$ is the identity function on $A$)

\[
f^{-1} \circ f = I_A
\]

|Domain| = |Codomain| = |Range|

*We can invert the function!!*
Graphs of Functions

• We can represent a function \( f:A \rightarrow B \) as a set of ordered pairs \( \{(a,f(a)) \mid a \in A\} \).

• Note that \( \forall a \), there is only 1 pair \( (a,b) \).
  • Later (ch.6): relations loosen this restriction.

• For functions over numbers, we can represent an ordered pair \( (x,y) \) as a point on a plane.
  • A function is then drawn as a curve (set of points), with only one \( y \) for each \( x \).
A Couple of Key Functions

• In discrete math, we will frequently use the following two functions over real numbers:

• **Def.** The *floor* function $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$, where $\lfloor x \rfloor$ ("floor of $x$") means the largest (most positive) integer $\leq x$. *Formally*, $\lfloor x \rfloor \equiv \max(\{ j \in \mathbb{Z} \mid j \leq x \})$.

• **Def.** The *ceiling* function $\lceil \cdot \rceil : \mathbb{R} \to \mathbb{Z}$, where $\lceil x \rceil$ ("ceiling of $x$") means the smallest (most negative) integer $\geq x$. *Formally*, $\lceil x \rceil \equiv \min(\{ j \in \mathbb{Z} \mid j \geq x \})$.
Visualizing Floor & Ceiling

• Real numbers “fall to their floor” or “rise to their ceiling.”

• Note that if \( x \notin \mathbb{Z} \),
  
  \[
  \lfloor -x \rfloor \neq - \lfloor x \rfloor \quad \text{and} \quad \lceil -x \rceil \neq - \lceil x \rceil
  \]

• Note that if \( x \in \mathbb{Z} \),
  
  \[
  \lfloor x \rfloor = \lceil x \rceil = x.
  \]
Plots with floor/ceiling: Example

- Plot of graph of function $f(x) = \lfloor x/3 \rfloor$:

Note, the open dot denotes that in the limit the point is not on the curve, and the closed dot means it is on the curve.
Operators (general definition)

- **Def.** An *n-ary operator over* (or *on*) the set $S$ is any function from the set of ordered $n$-tuples of elements of $S$, to $S$ itself.

- **Ex.** If $S=\{T,F\}$,
  - $\neg$ can be seen as a unary operator, and $\land, \lor$ are binary operators on $S$.

- **Ex.** $\cup$ and $\cap$ are binary operators on the set of all sets. (See HW3 with $\bigcup_{i=1}^{k}$ notation)
Combining Function Operators

• 
  
  \( +, \times \) ("plus", "times") are binary operators over \( \mathbb{R} \). (Normal addition & multiplication.)

• Therefore, we can also add and multiply functions

• Def. Let \( f, g: \mathbb{R} \to \mathbb{R} \).

  • \( (f + g): \mathbb{R} \to \mathbb{R} \), where \( (f + g)(x) = f(x) + g(x) \)

  • \( (f \times g): \mathbb{R} \to \mathbb{R} \), where \( (f \times g)(x) = f(x) \times g(x) \)
Function Composition Operator

- **Def.** Let \( g: A \rightarrow B \) and \( f: B \rightarrow C \). The *composition of \( f \) and \( g \)*, denoted by \( f \circ g \), is defined by \((f \circ g)(a) = f(g(a))\).

- **Remark.** \( \circ \) (like Cartesian product \( \times \), but unlike \(+, \wedge, \cup\)) is non-commuting. (Generally, \( f \circ g \neq g \circ f \).)
Review of §2.3 (Functions)

• Function variables \( f, g, h, \ldots \)
• Notations: \( f: A \rightarrow B, f(a), f(A) \).
• Terms: image, preimage, domain, codomain, range, one-to-one, onto, strictly (in/d)ecreasing, bijective, inverse, composition.
• Function unary operator \( f^{-1} \), binary operators +, -, etc., and ◦.
• The \( \mathbb{R} \rightarrow \mathbb{Z} \) functions \([x]\) and \([x]\).
A sequence or series is just like an ordered n-tuple, except:

- Each element in the series has an associated index number. E.g., \( (a_1, a_2, \ldots, a_k) \)
- A sequence or series may be infinite.

A string is a sequence of symbols from some finite alphabet. (e.g., words in a language)

A summation is a compact notation for the sum of all terms in a (possibly infinite) series.

\[
\sum_{i=j}^{k} a_i \equiv a_j + a_{j+1} + \ldots + a_k
\]
Sequences

• **Def.** A sequence or series \( \{a_n\} \) is identified with a *generating function* \( f: S \to A \) for some subset \( S \subseteq \mathbb{N} \) and for some set \( A \).
  
  • Often we have \( S = \mathbb{N} \) or \( S = \mathbb{Z}^+ = \mathbb{N} - \{0\} \).
Recognizing Sequences

- Sometimes, you’re given the first few terms of a sequence and you need to find the sequence’s **generating function**, which is a procedure to enumerate the sequence.

- Examples: What’s the next number?
  - 1,2,3,4,…
  - 1,3,5,7,9,…
  - 2,3,5,7,11,…
  - 5 (the 5th smallest number >0)
  - 11 (the 6th smallest odd number >0)
  - 13 (the 6th smallest prime number)
Sequence elements, \( a_n \)

**Def.** If \( f \) is a generating function for a series \( \{a_n\} \), then for \( n \in S \), the symbol \( a_n \) denotes \( f(n) \), also called *term n* of the sequence.

The *index* of \( a_n \) is \( n \). (Or, often \( i \) is used.)

A series is sometimes denoted by listing its first and/or last few elements, and using ellipsis (\( \ldots \) ) notation.

*E.g.*, \( \{a_n\} = 0, 1, 4, 9, 16, 25, \ldots \) is taken to mean

\[
\forall n \in \mathbb{N}, \ a_n = n^2.
\]
Sequence Examples

• Some authors write “the sequence $a_1, a_2, ...$” instead of $\{a_n\}$, to ensure that the set of indices is clear. Be careful: often notation leaves the indices ambiguous, but context makes it clear.

• Ex. An example of an infinite series:
  • Consider the series $\{a_n\} = a_1, a_2, ..., \text{ where } (\forall n \geq 1) a_n = f(n) = 1/n$.
  • Then, we have $\{a_n\} = 1, 1/2, 1/3, ...$
Switch to blackboard mode