Some network flow problems in urban road networks

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Outline of Lecture

• Transportation modes, and some basic statistics
• Characteristics of transportation networks
• Flows and costs
• Distribution of flows
  – Behavioral assumptions
  – Mathematical formulation and solution
  – Applications
### Vehicle Miles of Travel: by mode

(U.S., 1997, Pocket Guide to Transp.)

<table>
<thead>
<tr>
<th>Mode</th>
<th>Vehicle-miles (millions)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air Carriers</td>
<td>4,911</td>
</tr>
<tr>
<td>General Aviation</td>
<td>3,877</td>
</tr>
<tr>
<td>Passenger Cars</td>
<td>1,502,000</td>
</tr>
<tr>
<td>Trucks</td>
<td></td>
</tr>
<tr>
<td>Single Unit</td>
<td>66,800</td>
</tr>
<tr>
<td>Combination</td>
<td>124,500</td>
</tr>
<tr>
<td>Amtrak(RAIL)</td>
<td>288</td>
</tr>
</tbody>
</table>
### Passenger miles by mode

(U.S., 1997, Pocket Guide to Transp.)

<table>
<thead>
<tr>
<th>Mode</th>
<th>Passenger-miles (millions)</th>
<th>% SHARE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air Carriers</td>
<td>450,600</td>
<td>9.75%</td>
</tr>
<tr>
<td>General Aviation</td>
<td>12,500</td>
<td>0.27%</td>
</tr>
<tr>
<td>Passenger Cars</td>
<td>2,388,000</td>
<td>51.67%</td>
</tr>
<tr>
<td>Other vehicles</td>
<td>1,843,100</td>
<td>34.56%</td>
</tr>
<tr>
<td>Buses</td>
<td>144,900</td>
<td>3.14%</td>
</tr>
<tr>
<td>Rail</td>
<td>26,339</td>
<td>0.56%</td>
</tr>
<tr>
<td>Other</td>
<td>1,627</td>
<td>0.04%</td>
</tr>
</tbody>
</table>
## Fatalities by mode (1997, US)

<table>
<thead>
<tr>
<th>Mode</th>
<th># of fatalities</th>
<th># per million pas.-mile</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air</td>
<td>631</td>
<td>0.001363</td>
</tr>
<tr>
<td>Highway</td>
<td>42013</td>
<td>0.00993</td>
</tr>
<tr>
<td>Railroad</td>
<td>602</td>
<td>0.022856</td>
</tr>
<tr>
<td>Transit</td>
<td>275</td>
<td>0.001898</td>
</tr>
<tr>
<td>Waterborne</td>
<td>959</td>
<td>N/A</td>
</tr>
</tbody>
</table>
How to “grow” a transportation system:

pop. & economic growth, land use and demand/supply balance
An example: Beijing, China

Population: 5.6 million (1986) -> 10.8 million (2000)
GDP: ~9-10% annual growth

Changes in land use

Changes in the highway network
The four step planning process

Activity pattern and forecast

Trip Generation

Trip Distribution

Modal Split

Trip Assignment

Link Flow

NEED FEEDBACK
EXAMPLE 1: HIGHWAY TRANSPORTATION
EXAMPLE 2: RAIL (SUBWAY) TRANSPORTATION

London

Stockholm
EXAMPLE 3: AIR TRANSPORTATION
TRANSPORTATION NETWORKS
AND THEIR REPRESENTATIONS

• Nodes (vertices) for connecting points
  – Flow conservation, capacity and delay

• Links (arcs, edges) for routes
  – Capacity, cost (travel time), flow propagation

• Degree of a node, path and connectedness

• A node-node adjacency or node-link incidence matrix for network structure
Characteristics of transportation networks

• Highway networks
  – Nodes rarely have degrees higher than 4
  – Many node pairs are connected by multiple paths
  – Usually the number of nodes < number of links < number of paths in a highway network

• Air route networks
  – Some nodes have much higher degrees than others (most nodes have degree one)
  – Many node pairs are connected by a unique path

• Urban rail networks
  – Falls between highway and air networks
Flows in a Highway Network

$N$: set of nodes

$A$: set of links

$I$: set of origins

$J$: set of destinations

$R_{ij}$: set of paths from origin $i$ to destination $j$

$t_a(v_a, C_a)$: link travel cost function

$q_{ij}$: Traffic demand from origin $i$ to destination $j$

$C_a$: Capacity on link $a$
Flows in a Highway Network (Cont’d)

- Path flows: \( \{ f_r^{ij}, r \in R_{ij}, i \in I, j \in J \} \)
  - Flow conservation equations
    \[
    \sum_{r \in R_{ij}} f_r^{ij} = q_{ij}, i \in I, j \in J
    \]
    \[
    f_r^{ij} \geq 0
    \]
  - Set of feasible path flows
    \[
    S = \left\{ f = (\ldots, f_r^{ij}, \ldots) \mid \sum_{r \in R_{ij}} f_r^{ij} = q_{ij}; f_r^{ij} \geq 0; r \in R_{ij}, i \in I, j \in J \right\}
    \]
Flows in a Highway Network (Cont’d)

• Origin based link flows: $\{v^i_a, a \in A, i \in I\}$
  
  – Flow conservation equations

  \[
  \sum_{a \in A^+_n} v^i_a = \sum_{j \in J} q_{ij}, i \in I \\
  \sum_{a \in A^-_n} v^i_a = q_{ij}, j \in J \\
  \sum_{a \in A^+_n} v^i_a - \sum_{a \in A^-_n} v^i_a = 0, n \in N \setminus (I \cup J) \\
  v^i_a \geq 0, i \in I, a \in A
  \]

  $A^-_n = \{\text{all links entering node } n\}, n \in N$

  $A^+_n = \{\text{all links leaving node } n\}, n \in N$

  – Set of feasible origin based link flows

  $S' = \left\{ v^i = (\cdots, v^i_a, \cdots) \mid \{v^i_a, i \in I, a \in A\} \text{satisfies the above equations} \right\}$
Flows in a Highway Network (Cont’d)

• **Link flows:** \( \nu = (\ldots, \nu_a, \ldots)^T \)
  
  – Set of feasible link flows

\[
\Omega = \left\{ \nu \mid \nu_a = \sum_{r \in I} \sum_{j \in J} \sum_{r \in R_{ij}} f_r^{ij} \delta_{ar}^{ij}, a \in A; \sum_{r \in R_{ij}} f_r^{ij} = q_{ij}; f_r^{ij} \geq 0; r \in R_{ij}, i \in I, j \in J \right\}
\]

where

\[
\delta_{ar}^{ij} = \begin{cases} 
1, & \text{if path } r \in R_{ij} \text{ using link } a \\
0, & \text{otherwise}
\end{cases}
\]

It is a convex, closed and bounded set
Costs in a Highway Network (Cont’d)

• Travel cost on a path \( r \in R_{ij}, i \in I, j \in J \)

\[
c_r^{ij} = \sum_{a \in A} t_a(v_a) \delta^{ij}_{ar}, r \in R_{ij}, i \in I, j \in J
\]

• The shortest path from origin \( i \) to destination \( j \)

\[
\mu_{ij} = \min_{r \in R_{ij}} \{c_r^{ij}\}, i \in I, j \in J
\]

• Total system travel cost

\[
\sum_{a \in A} t_a(v_a)v_a
\]
Behavioral Assumptions

Act on self interests (User Equilibrium):

• Travelers have full knowledge of the network and its traffic conditions
• Each traveler minimizes his/her own travel cost (time)

Act on public interests (System Optimal):

• Travelers choose routes to make the total travel time of all travelers minimal (which can be achieved through choosing the routes with minimal marginal travel cost)

\[
\min \sum_{a \in A} t_a(v_a)v_a
\]
THE USER EQUILIBRIUM CONDITION

• At UE, no traveler can unilaterally change his/her route to shorten his/her travel time (Wardrop, 1952). It’s a Nash Equilibrium. Or

• At UE, all paths connecting an origin-destination pair that carry flow must have minimal and equal travel time for that O-D pair

\[ f_r^{ij} (c_r^{ij} - \mu_{ij}) = 0, \quad (c_r^{ij} - \mu_{ij}) \geq 0, \quad f_r^{ij} \geq 0 \]

• However, the total travel time for all travelers may not be the minimum possible under UE.
A special case: no congestion, infinite capacity

- Travel time is independent of flow intensity
- UE & SO both predict that all travelers will travel on the shortest path(s)
- The UE and SO flow patterns are the same

\( \{f_{r, i}^{ij}, r \in R_{ij}, i \in I, j \in J\} \)

We can check the UE and SO conditions

<table>
<thead>
<tr>
<th>i=1, j=2, q_{12}=12</th>
<th>i=1, j=2, q_{12}=12</th>
</tr>
</thead>
<tbody>
<tr>
<td>t1=1</td>
<td>V1=12, f1=12, ( \mu )=1</td>
</tr>
<tr>
<td>t2=10</td>
<td>V2=0, f2=0</td>
</tr>
<tr>
<td>t3=40</td>
<td>V3=0, f3=0</td>
</tr>
</tbody>
</table>

We can check the UE and SO conditions
A case with congestion

Link cost functions:
\[ t_1(v) = 1 + v_1 \]
\[ t_2(v) = 1 + v_2 + \frac{1}{2} v_1 \]
\[ t_3(v) = 40 \]

O-D demand:
\[ q_{12} = 12 \]

UE path flow pattern:
\[ f_{12}^{12*} = 8, f_{22}^{12*} = 4, f_{32}^{12*} = 0 \]

Path travel cost pattern:
\[ c_{12}^{12*} = 9, c_{22}^{12*} = 9, c_{32}^{12*} = 40 \]

UE origin based link flow pattern
\[ v_1^{1*} = 8.0, v_2^{1*} = 4.0, v_3^{1*} = 0.0 \]

UE O-D travel cost:
\[ \mu_{12} = 9 \]

UE link flow pattern:
\[ v_1^* = 8, v_2^* = 4, v_3^* = 0 \]

\[ \sum_i v_i t_i = 108 \]

SO:
\[ \min v_1 t_1(v) + v_2 t_2(v) + v_3 t_3(v) \]
\[ v_1 + v_2 + v_3 = 12 \]
\[ v_1, v_2, v_3 \geq 0 \]

\[ v_1 = 6, t_1 = 7 \]
\[ v_2 = 6, t_2 = 10 \]
\[ v_3 = 0, t_3 = 40 \]
\[ \sum_i v_i t_i = 102 \]
The Braess’ Paradox

\[ q_1 = 6 \quad q_2 = 6 \]

2 paths

\[ \text{flow:} \]

\[ f_1 = 3 \quad f_2 = 3 \]

\[ + = 6 \]

\[ t_1 = 50 + x_1 \]
\[ t_2 = 50 + x_2 \]
\[ t_3 = 10 x_3 \]
\[ t_4 = 10 x_4 \]

\[ t_1 + t_4 = 53 + 30 = 83 \]
\[ t_3 + t_2 = 30 + 53 = 83 \]

Total travel time \[ \sum x_i t_i = 166 \]

Now, if we add an additional link to the network with the following cost function \[ t_5 = 10 + x_5 \]
The Braess’ Paradox-Cont.

Now, if we add an additional link to the network with the following cost function $t_5 = 10 + x_5$

Now we have a new path $3 \rightarrow 5 \rightarrow 4$ with flow

so $c_3 = t_3 + t_5 + t_4 = 0 + 10 + 0 = 10$

$< c_2$ & $c_1$

traffic is no longer in equilibrium $\rightarrow$ new equilibrium will be produced.
The Braess’ Paradox—Cont.

By inspection, we shift 1 unit of flow from path 1 & 2, resp. to path 3

\[
\begin{align*}
\text{path} & \quad \text{1} \rightarrow \text{2} \rightarrow \text{4} \\
& \quad x_1 = 2 \\
& \quad x_2 = 2 \\
& \quad x_3 = 4 \\
\text{path} & \quad \text{2} \rightarrow \text{3} \rightarrow \text{2} \\
& \quad x_4 = 4 \\
\end{align*}
\]

\[
c_1 = t_1 + t_4 = 52 + 40 = 92
\]
\[
c_2 = t_2 + t_3 = 52 + 40 = 92
\]
\[
c_3 = t_3 + t_5 + t_4 = 40 + 12 + 40 = 92
\]

a new UE state

but the path travel times are higher (92 vs. 83)

\[
\text{total travel time} = \sum t_i \cdot x_i = 276 > 166
\]
General Cases for UE:

\[
\min_v h(v) = \sum_{a \in A} \int_0^{\chi_a} t_a(\omega) d\omega
\]

subject to

\[
\sum_{r \in R_{ij}} f_r^{ij} = q_{ij}, i \in I, j \in J
\]

\[
v_a = \sum_{i \in I} \sum_{j \in J} \sum_{r \in R_{ij}} f_r^{ij} \delta_{ar}, a \in A
\]

\[
f_r^{ij} \geq 0, r \in R_{ij}, i \in I, j \in J
\]

With an increasing travel time function, this is a strictly (nonlinear) convex minimization problem.

It can be shown that the KKT condition of the above problem gives precisely the UE condition
The relation between UE and SO

\[ \min_{v \in \Omega} h(v) = \sum_{a \in A} \int_{0}^{\nu_a} t_a(\omega) d\omega \]

\[ \hat{t}_a(\nu_a) = t_a(\nu_a) + \nu_a \frac{dt_a}{d\nu_a} \]

\[ \min_{v \in \Omega} \hat{h}(v) = \sum_{a \in A} \int_{0}^{\nu_a} \hat{t}_a(\omega) d\omega \]

\[ \min_{v \in \Omega} H(v) = \sum_{a \in A} \nu_a t_a(\nu_a) \]

\[ \tilde{t}_a(\nu_a) = \frac{1}{\nu_a} \int_{0}^{\nu_a} t_a(\omega) d\omega \]

\[ \min_{v \in \Omega} \tilde{H}(v) = \sum_{a \in A} \nu_a \tilde{t}_a(\nu_a) \]
Algorithms for Solving the UE Problem

- Generic numerical iterative algorithmic framework for a minimization problem

Yes

No
Algorithms for Solving the UE Problem (Cont’d)
Applications

• Problems with thousands of nodes and links can be routinely solved
• A wide variety of applications for the UE problem
  – Traffic impact study
  – Development of future transportation plans
  – Emission and air quality studies
Intelligent Transportation System (ITS) Technologies

• On the road
• Inside the vehicle
• In the control room
"EYES" OF THE ROAD

- Loop detector
- Infrared detector
- Ultrasonic detector
- Video detector
SMART ROADS
SMART VEHICLES

SAFETY, TRAVEL SMART GAGETS, MOBILE OFFICE(?)
SMART PUBLIC TRANSIT

• GPS + COMMUNICATIONS FOR
  – BETTER SCHEDULING & ON-TIME SERVICE
  – INCREASED RELIABILITY

• COLLISION AVOIDANCE FOR
  – INCREASED SAFETY
SMART CONTROL ROOM
If you wish to learn more about urban traffic problems

- ECI 256: Urban Congestion and Control (every Fall)
- ECI 257: Flows in Transportation Networks (Winter)