Game Theory & Networks

(an incredibly brief overview)

Andrew Smith

ECS 253/MAE 289

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Game theory can help us answer important questions for scenarios where:

players/agents (nodes) are *autonomous and selfish*, and

player’s connections (edges) directly affect their utility.

Terminology for Games on Networks:

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  (e.g.: *cooperate* or *defect* from Prisoner’s Dilemma)

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- **Utility**: $U_i(S)\ \forall\ i \in N$; how much benefit a player *i* gets from strategy *S*.

Nash Equilibrium

Pure-strategy Nash equilibrium: A *pure strategy* for each player, such that, given the strategy of the other players, no player would do better playing a different strategy.

	Cooperate	Defect
Cooperate	-1,-1	-10,0
Defect	0,-10	-5,-5

Prisoner's Dilemma

Nash Equilibrium

Mixed-strategy Nash equilibrium: A *mixed strategy* for each player, such that, given the strategy of the other players, no player would do better by changing their strategy.

	Swerve	Straight
Swerve	0,0	-1,1
Straight	1,-1	-10,-10

Chicken

Two “Unfair” Pure-Strategy
Nash Equilibria!

Mixed-Strategy Nash Equilibrium

- Player 2 chooses swerve with probability *p* and straight with probability *1-p.*

	p	*1-p*
	Swerve	Straight
Swerve	0,0	-1,1
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Mixed-Strategy Nash Equilibrium

- Player 2 chooses swerve with probability *p* and straight with probability *1-p.*
- Player 2 wishes to make Player 1 *indifferent* about what strategy to choose

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Mixed-Strategy Nash Equilibrium

- Player 2 chooses swerve with probability *p* and straight with probability *1-p.*
- Player 2 wishes to make Player 1 *indifferent* about what strategy to choose (i.e., *maximize expected payoff*).

	p Swerve	*1-p* Straight
Swerve	0,0	-1,1
Straight	1,-1	-10,-10

Chicken

$u_1(Swerve) = u_1(Straight)$

0*p + -1*(1-p) = 1*p + -10*(1-p)

p-1=11p-10

p=9/10

Mixed-Strategy Nash Equilibrium

- Now, Player 1 must also randomize (making Player 2 indifferent)

$u_2(Swerve) = u_2(Straight)$

0*q + -1*(1-q) = 1*q + -10*(1-q)
q-1=11q-10
q=9/10

		p=9/10	1-p=1/10
		Swerve	Straight
q	Swerve	0,0	-1,1
1-q	Straight	1,-1	-10,-10

Chicken

Mixed-Strategy Nash Equilibrium

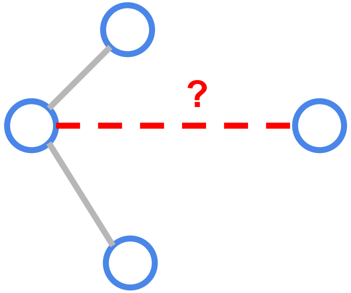
- Now, Player 1 must also randomize (making Player 2 indifferent)
- **Mixed-strategy Nash equilibria= (9/10,1/10),(9/10,1/10)**

		p=9/10	*1-p=1/10*
		Swerve	Straight
q=9/10	Swerve	0,0	-1,1
1-q=1/10	Straight	1,-1	-10,-10

Chicken

The most well studied network scenarios....

Network Formation Games



How do networks form given selfish, utility-driven players?

Social networks, supply networks, power grids, etc.

The most well studied network scenarios....

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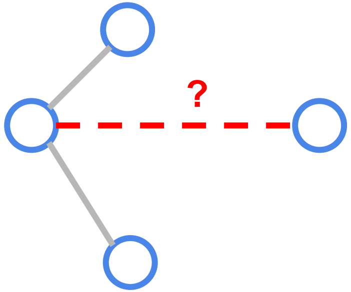
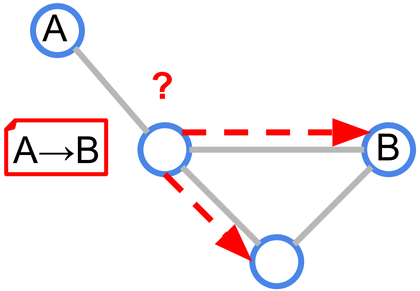


Routing Games



How reliable or efficient is the routing of flow given a network structure (and selfish players)?

Packet routing, traffic flow,information dissemination

The most well studied network scenarios....

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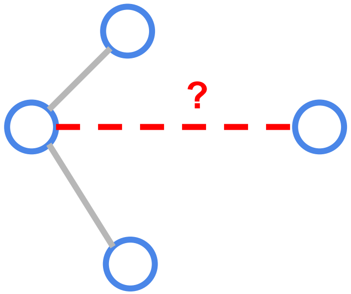
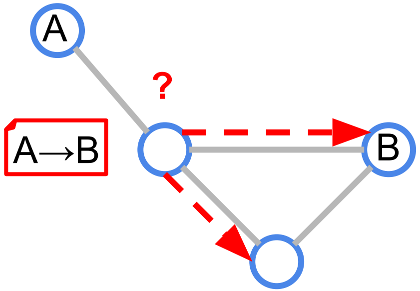


Routing Games

Equilibria in "Routing Games" can usually be illustrated by Pigou's Principle



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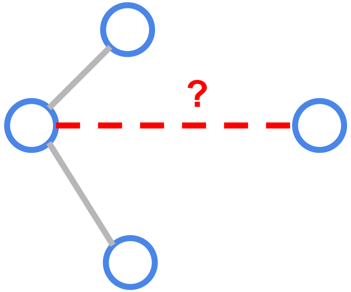
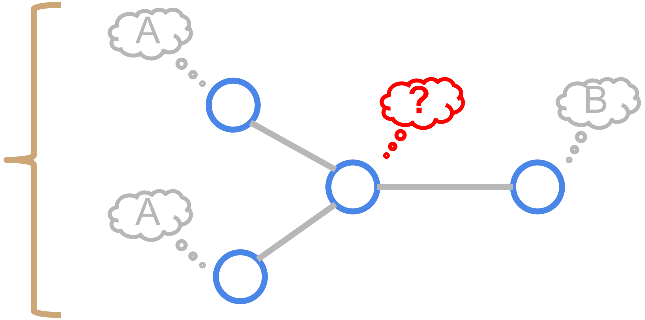


How do opinions/ideas/ diseases spread in a network?

Epidemic spread, voting, technology adaptation

Opinion Dynamics



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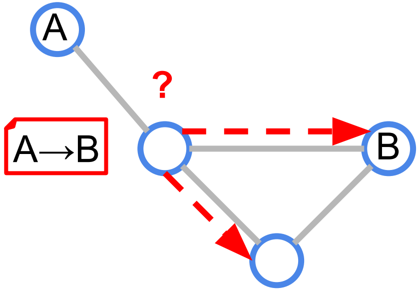

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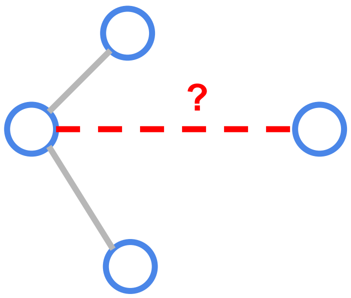

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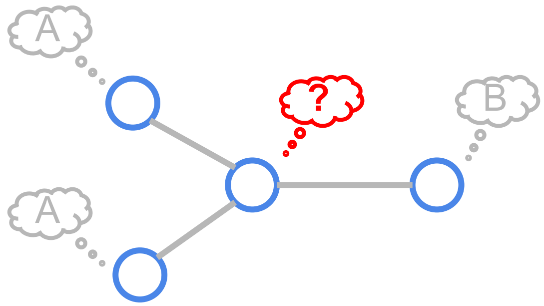

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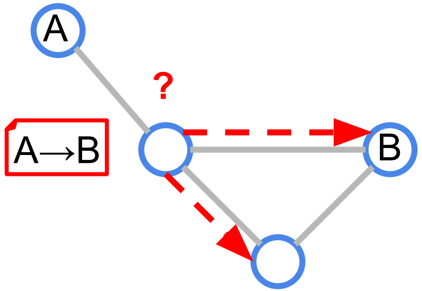

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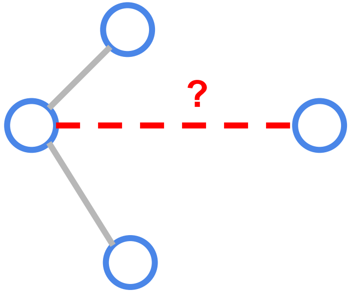
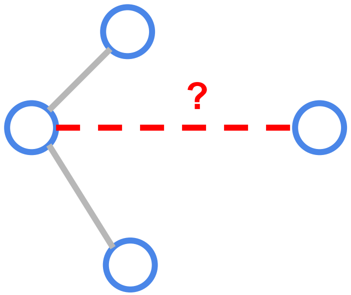


- **Scenario:** *N* players would like to increase their utility by creating edges with each other (but not if it's too costly!)

Network Formation Games



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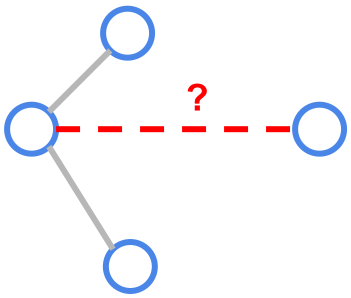

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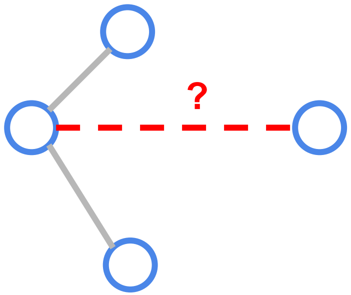
Actions for player *i* (for all i):

{**don't build edge**,**build edge**}$^{N-1}$

Network Formation Games



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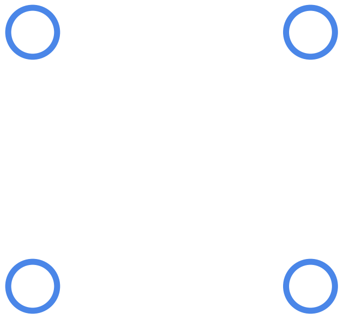
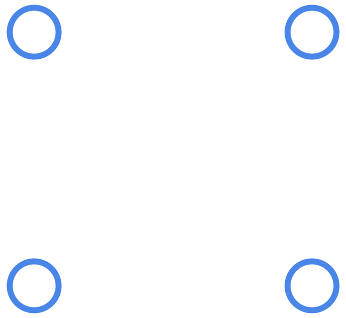
- **Question:** What networks emerge in Nash equilibria?

Network Formation Games



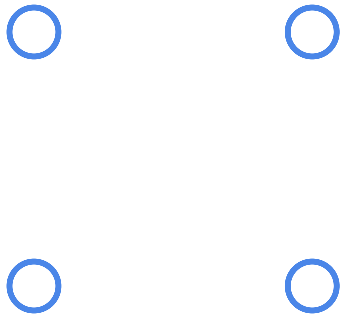
- 4 players/nodes (N=4); empty network

Network Formation Games



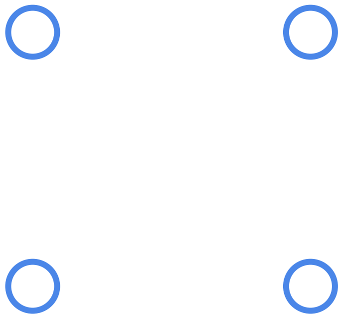
- 4 players/nodes (N=4); empty network
- Does any *one* player want to deviate from the current **strategy**?

Network Formation Games



- 4 players/nodes (N=4); empty network
- Does any one player want to deviate from the current **strategy**?
  - No! -- They couldn't if they tried.
- **Mutual edge creation** makes Nash equilibria less interesting...

Network Formation Games



- A network is **pairwise stable** if there is no other network configuration such that:
  - Any two pairs of nodes wishes to add an edge, and...
  - Any one node wishes to remove an edge.
- Now, we care about the *utilities of players.*

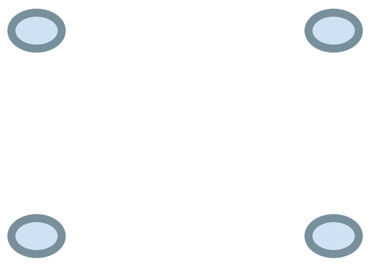
Symmetric Connections Model

Distance-based utility function

$$u_i = b(\ell_{ij}) - d_i c$$

$b(\ell_{ij})$ = some function on the shortest path between player *i* and player *j*.

A game with 4 players/nodes



Jackson, M.O., 2005. A survey of network formation models: stability and efficiency. *Group Formation in Economics: Networks, Clubs, and Coalitions*.

Symmetric Connections Model

Distance-based utility function

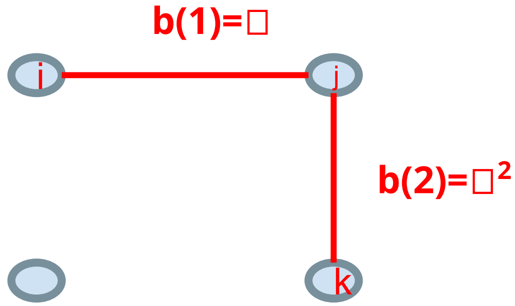
$$u_i = b(\ell_{ij}) - d_i c$$

$b(\ell_{ij})$ = some function on the shortest path between player *i* and player *j*.

d_i = total degree of player *i*.

We will assume $b(k) = □^k$ (for $□ < 1$)

A game with 4 players/nodes

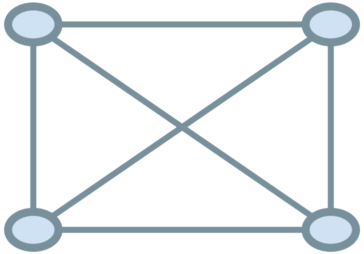


$$\mathbf{u_i = □ + □^2 - c}$$

$$\mathbf{u_j = □ + □ - 2c}$$

$$\mathbf{u_k = □ + □^2 - c}$$

Pairwise Stability in Symmetric Connections Model



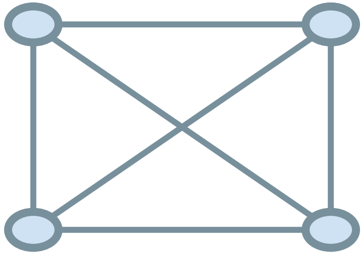
$c < b(1) - b(2)$

A complete network!

$b(1) < c$

$b(1) - b(2) < c < b(1)$

Pairwise Stability in Symmetric Connections Model



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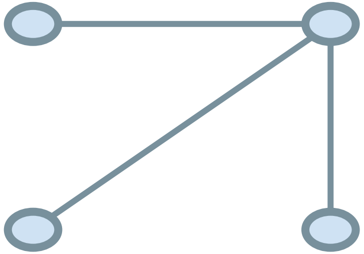

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A star! (and possibly others)

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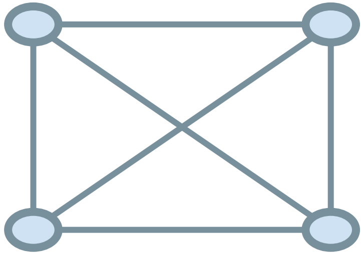
Pairwise Stability in Symmetric Connections Model



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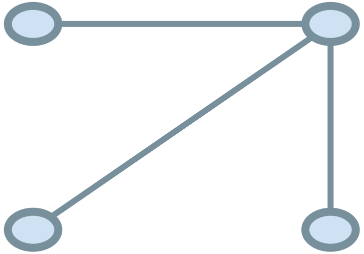

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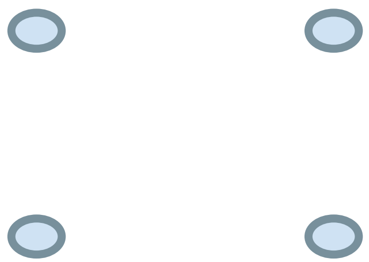
The empty network!

Efficient Solutions in Symmetric Connections Model

Consider the case when cost is relatively high...

A game with 4 players/nodes



$b(1) < c$

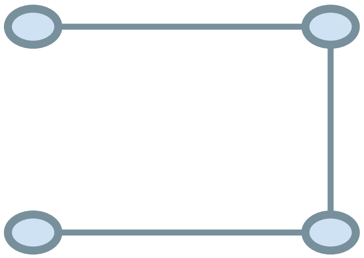
The empty network!
Each player gets nothing!

Efficient Solutions in Symmetric Connections Model

Consider the case when cost is relatively high...

- A *path* through all nodes is better for everyone!

A game with 4 players/nodes



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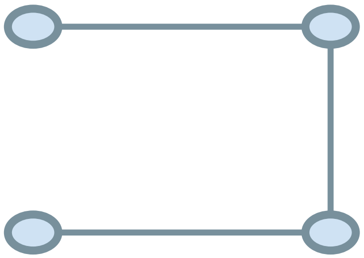
Efficient Solutions in Symmetric Connections Model

Consider the case when cost is relatively high…

- A *path* through all nodes is better for everyone!
- **Efficient** solutions maximize the sum of all players' utility

A game with 4 players/nodes



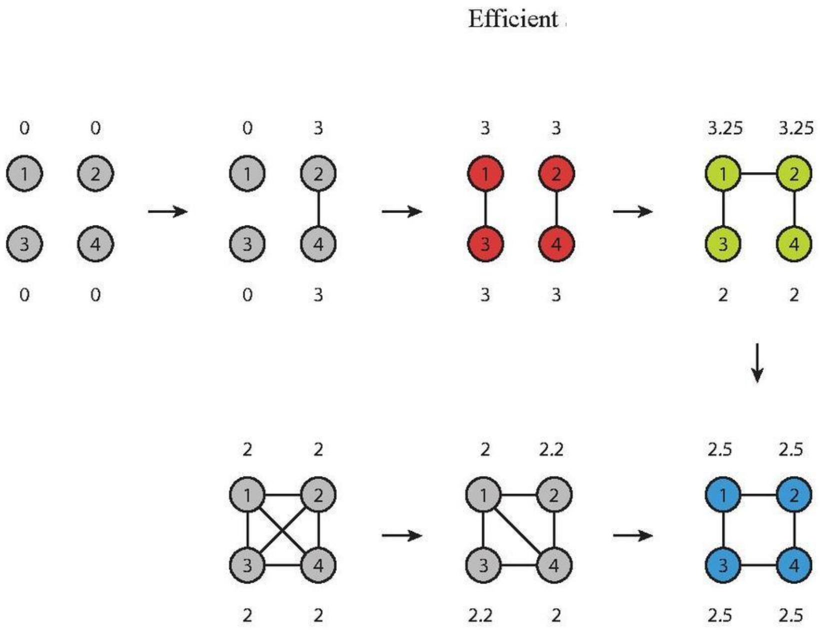
b(1) < c

Efficient!
(Given c <= b(1) + b(2))

Solution concepts in network games

Other solutions (besides NE) can also be desired:

- **Efficient strategy:** maximizes the sum of players' utility

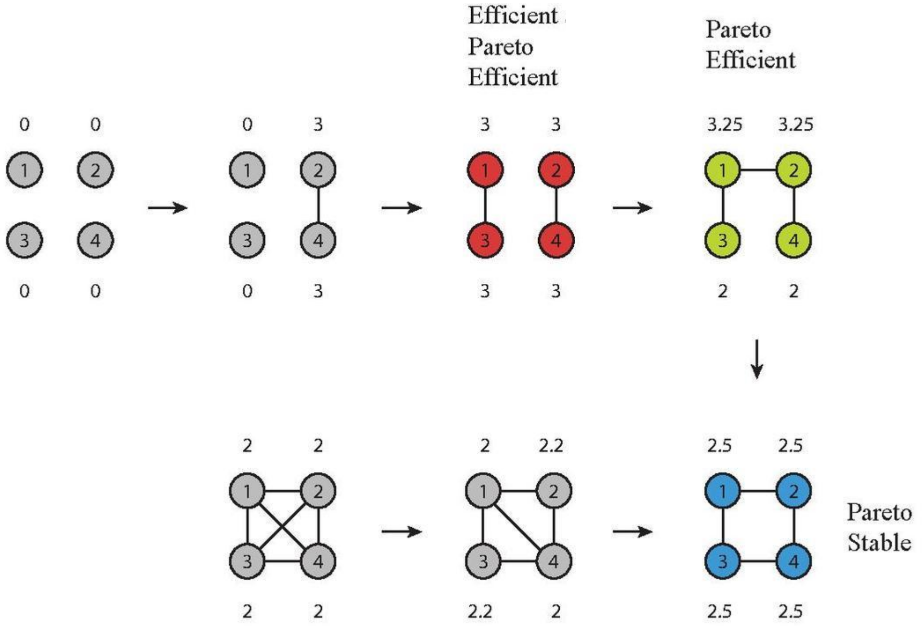


Source: https://en.wikipedia.org/wiki/Strategic_Network_Formation

Solution concepts in network games

Other solutions (besides NE) can also be desired:

- **Efficient strategy**: maximizes the sum of players' utility
- **Pareto optimal** (or pareto efficient): network such that there **is no other network *g'*** where: $u_i(g') >= u_i(g)$ *for all i* and $u_i(g') > u_i(g)$ *for at least 1 i*.



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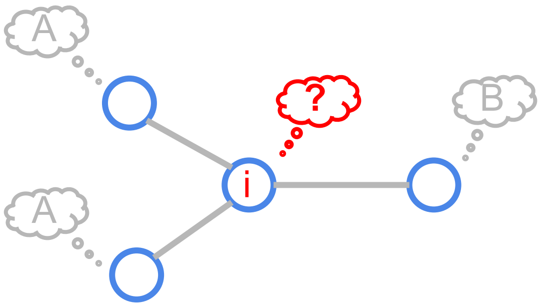
Opinion Dynamics via "the Majority Game"

Majority Game:

- *N* players/nodes
- *A* = {A,B}

- The set of neighbors of player *i* who believe A: $\mathbf{N_i(A)}$



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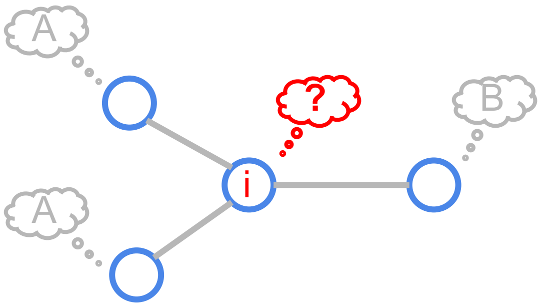
- The set of neighbors of player *i* who believe A: $\mathbf{N_i(A)}$
- **Majority utility function:**
    - If $|\mathbf{N_i(A)}| > \frac{1}{2} * deg(i)$, $u_i(A) > u_i(B)$
    - Otherwise, $u_i(B) > u_i(A)$



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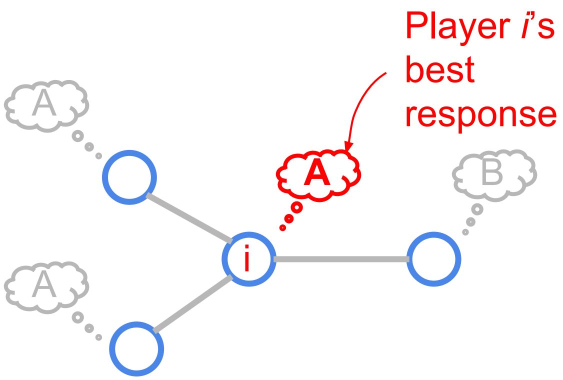
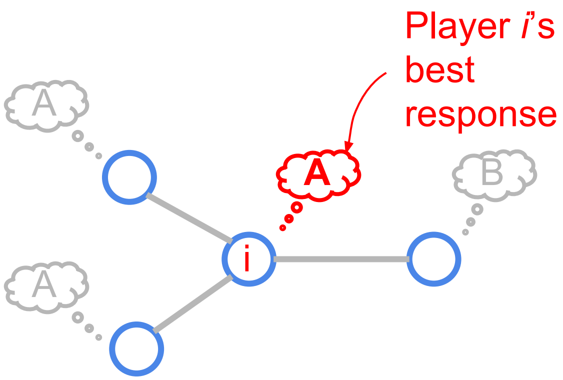


NE in “the Majority Game”

- Nash equilibrium: When no player wishes to change their belief, given the other players’ beliefs.



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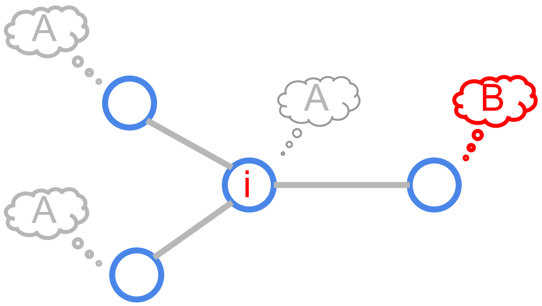
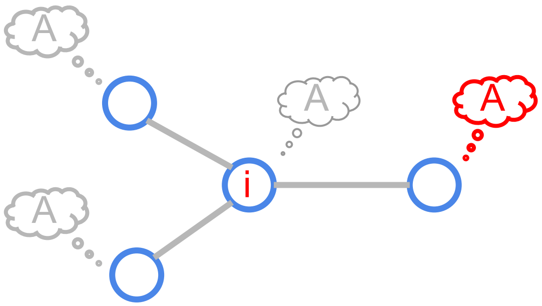


This is not a Nash equilibrium!

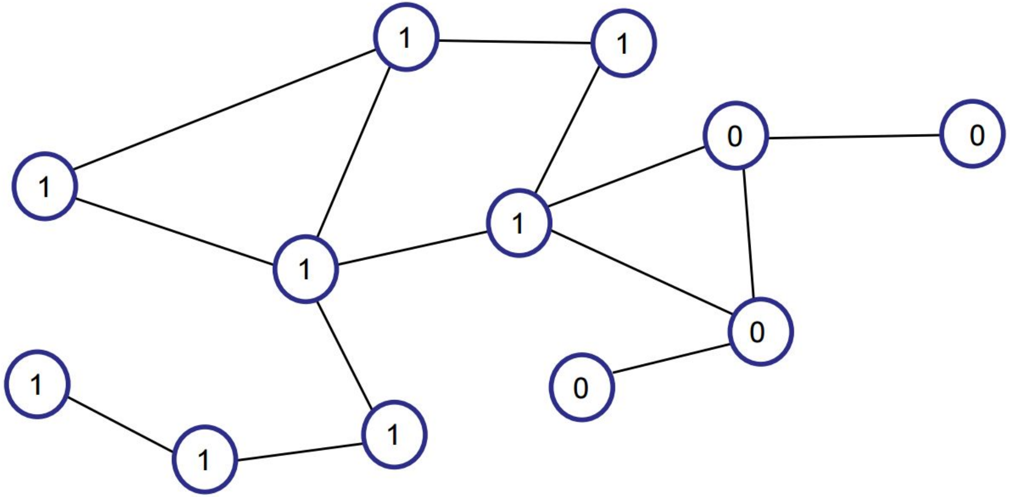
NE in “the Majority Game”

- Nash equilibrium: When no player wishes to change their belief, given the other players’ beliefs.
- Generally, every player choosing A and every player choosing B is a NE.
    - But there can be others...



This IS a Nash equilibrium!

NE in “the Majority Game”



The initial configuration matters: flipping everyone’s opinion is also stable!

(source: Jackson, M., **Games on Networks**, Handbook of Game Theory, Vol. 4, 2014.)

Extensions of “the Majority Game”

- **Coordination games:** Highest utility is gained by coordinating with neighbors; miscoordination incurs a cost. What thresholds and

	A	B
A	(b,b)	(-c,0)
B	(0,-c)	(0,0)

- **Stability analysis of equilibria:** Which equilibria are most stable to a player “changing their mind”?
- **Resources:**
  - Jackson, M.O. and Zenou, Y., 2014. **Games on networks**. *Handbook of game theory*,.
  - Kearns, M., 2007. **Graphical Games.** *Algorithmic Game Theory*.

Final notes

- Many network-based games can be modeled as evolutionary processes:
  - **Network formation:** Start with an initial network, and add/remove edges until no player wishes to deviate (NE found).
  - **Opinion dynamics:** Seed beliefs randomly (or empirically), and update players' beliefs until no player wishes to change their belief (NE found).

- *Algorithmic Game Theory,* Noam Nisan, Tim Roughgarden et. al
- *Social and Economic Networks,* Matthew Jackson.